Adaptive Mixed Finite Element Approximations of Distributed Optimal Control Problems for Elliptic Partial Differential Equations

Dissertation

zur Erlangung des akademischen Titels
eines Doktors der Naturwissenschaften
der Mathematisch-Naturwissenschaftlichen
Fakultät der Universität Augsburg

vorgelegt von Meiyu Qi
geboren am 11.03.1983 in Tianjin, China

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Mündliche Prüfung: 09.August 2011

Acknowledgments

This work would not have been realized without the most valuable assistance and support of various people and organizations.

First of all, I would like to express my sincere thanks to my advisor, Prof.Dr. Ronald H.W. Hoppe, for his continuous encouragement and for numerous discussions on the topic of my dissertation.

Secondly, I am thankful to Prof.Dr. Malte Peter who agreed to act as a referee.

My thanks also go to Dr. Yuri Iliash for his tremendous help in the implementation of the adaptive code.

This dissertation has been supported by a grant from the German Academic Exchange Service (DAAD).

Last but not least, my special thanks go to my family which always did believe in me.

Contents

Abstract

We consider adaptive mixed finite element methods (AMFEM) for unconstrained optimal control problems associated with linear second order elliptic boundary value problems featuring distributed controls and a quadratic tracking-type objective functional. The focus is on solvers for the associated optimality system and on residual-type a posteriori error estimators for adaptive refinement of the underlying simplicial triangulations of the computational domain. In particular, for the numerical solution of the mixed finite element discretized optimality system we use preconditioned Richardson-type iterations with preconditioners that can be constructed by means of appropriately chosen left and right transforms. The residual a posteriori error estimators can be derived within the framework of a unified a posteriori error control which facilitates the proof of its reliability by evaluating the residuals in the respective dual norms. Numerical results illustrate the performance of the AMFEM.

Chapter 1

Introduction

In this contribution, we study adaptive mixed finite element approximations of unconstrained optimally controlled boundary value problems for linear second order elliptic partial differential equations with distributed controls based on simplicial triangulations of the computational domain.

The efficient numerical solution of boundary value problems for elliptic PDE and systems thereof by adaptive finite element methods is well documented in the literature. We refer to the monographs [1, 4, 6, 24, 54, 61] and the references therein. Among several error concepts that have been developed over the past decades there are residual-type estimators [1, 4, 61] that rely on the appropriate evaluation of the residual in a dual norm, hierarchical type estimators [4] where the error equation is solved locally using higher order elements, error estimators that are based on local averaging [16, 66], the so-called goal oriented dual weighted approach [6, 24] where information about the error is extracted from the solution of the dual problem, and functional type error majorants [54] that provide guaranteed sharp upper bounds for the error. A systematic comparison of the performance of these estimators for a basic linear second order elliptic PDE has been provided recently in [19].

A systematic mathematical treatment of optimally controlled elliptic PDE including existence and uniqueness results as well as the derivation of necessary and sufficient optimality conditions can be found in the seminal monograph [50] and the more recent textbooks [25, 31, 38, 49, 60]. As far as the a posteriori error analysis of adaptive finite element schemes for PDE constrained optimal control problems is concerned, for optimally controlled elliptic problems classical residual-based error estimators have been derived in [26, 27, 32, 36, 37, 39, 40, 41, 46, 47], whereas the goal-oriented dual weighted approach has been applied in [7, 8, 33, 34, 35, 62, 65]. With regard to other available techniques we note that hierarchical estimators have been considered in [9], those based on local averaging in [48], and those using functional type error majorants in [28, 29]. For further references, we refer to the recent mono-

graph [53].
Although mixed finite element discretizations of elliptic PDE have been studied extensively (cf., e.g., [13] and the references therein) including the development, analysis, and implementation of a posteriori error estimates [2, 11, 15, 64], there is only little work on its application to optimally controlled elliptic boundary value problems within an adaptive framework [21, 51].

Adaptive finite element methods for optimal control problems associated with PDE consist of successive loops of the cycle

$$\text{SOLVE} \quad \Longrightarrow \quad \text{ESTIMATE} \quad \Longrightarrow \quad \text{MARK} \quad \Longrightarrow \quad \text{REFINE}\,.$$

Here, SOLVE stands for the numerical solution of the discretized optimality system. The step ESTIMATE is devoted to the derivation of an a posteriori error estimator whose contributions are used for the realization of adaptivity in space. The subsequent step MARK deals with the selection of elements and edges of the triangulation for refinement based on the information provided by the local contributions of the a posteriori error estimator. We will use the bulk criterion from [22], meanwhile also known as Dörfler marking. The final step REFINE addresses the technical realization of the refinement process. In particular, refinement will be based on newest vertex bisection (cf., e.g., [5, 20, 56]).

The novelty of the adaptive mixed finite element approximation in this contribution is twofold. Firstly, as far as the step SOLVE of the adaptive cycle is concerned, we will solve the resulting block-structured saddle point problem numerically by a preconditioned Richardson-type iteration with a preconditioner derived from suitable left and right transforms. We note that transforming iterations have been used as smoothers within multigrid methods [63] as well as for the iterative solution of KKT systems in PDE constrained optimization [42, 43, 44, 57, 58]. Secondly, the second step ESTIMATE features a residual-type a posteriori error estimator which can be derived and analyzed within the framework of unified a posteriori error control [17].

The paper is organized as follows: In chapter 2, we provide basic functional analytic notations (subsection 2.1) and then consider an unconstrained elliptic optimal control problem with a tracking type objective functional and distributed controls (section 2.2) including the first order optimality conditions (Theorem 2.2 in section 2.3). Chapter 3 is devoted to the primal mixed formulation of the optimality system which results from a reformulation of the second order elliptic equation as a first order system. Its operator theoretic formulation gives rise to a bounded linear, bijective operator which implies unique solvability of the optimality system as well as continuous dependence on the data (Theorem 3.2). Moreover, given any conforming approximation of the optimality system, the error can be bounded from above in terms of associated residuals (Corollary 3.3). In chapter 4, we deal with the mixed finite element approximations of the optimality system by means of the lowest order Raviart-Thomas elements with respect to a shape regular family of simplicial triangulations of the computational domain. Algebraically, this gives rise to a block-structured linear algebraic system of saddle point type. The numerical solution of that saddle point problem by a preconditioned Richardson-type iteration is addressed in chapter 5. In particular, we present Uzawa-type preconditioners which can be derived by appropriately chosen left and right transforms. The following chapter 6 is concerned with the derivation of a residual-type a posteriori error estimator and the proof of its reliability (Theorem 6.1). For three representative examples, chapter 7 contains a documentation of numerical results illustrating the performance of the adaptive approach. Some concluding remarks are given in the final chapter 8.

Chapter 2

Optimally controlled elliptic problems

We consider optimally controlled linear second order elliptic PDE with a quadratic tracking type objective functional and distributed controls. In this contribution, we only study the unconstrained case, i.e., constraints are neither imposed on the control nor on the state.

2.1 Notations and preliminaries

We use standard notation from Lebesgue and Sobolev space theory [59]. In particular, given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d, d \in \mathbb{N}$, with boundary $\Gamma := \partial\Omega$, for $D \subseteq \Omega$. We refer to $L^p(D), 1 \leq p \leq \infty$ as the Banach spaces of p-th power integrable functions ($p < \infty$) and essentially bounded functions ($p = \infty$) on D with norm $\|\cdot\|_{L^p(D)}$. We denote by $L^p(D)_+$ the positive cone in $L^p(D)$, i.e., $L^p(D)_+ := \{v \in L^p(D) \mid v \geq 0 \text{ a.e. in } D\}$. In case $p = 2$, the space $L^2(D)$ is a Hilbert space whose inner product and norm will be referred to as $(\cdot,\cdot)_{L^2(D)}$ and $\|\cdot\|_{L^2(D)}$. For $m \in \mathbb{N}_0$, we denote by $W^{m,p}(D)$ the Sobolev spaces with norms

$$\|v\|_{W^{m,p}(D)} := \begin{cases} \Big(\sum\limits_{|\alpha| \leq m} \|D^\alpha v\|^p_{L^p(D)} \Big)^{1/p} & , \text{ if } p < \infty \\ \max\limits_{|\alpha| \leq m} \|D^\alpha v\|_{L^\infty(D)} & , \text{ if } p = \infty \end{cases},$$

where $\alpha = (\alpha_1, \cdots, \alpha_d)^T \in \mathbb{N}_0$ with $|\alpha| := \sum_{i=1}^d \alpha_i$, and refer to $|\cdot|_{W^{m,p}(D)}$ as the associated seminorms. For $p < \infty$ and $s \in \mathbb{R}_+, s = m + \sigma, m \in \mathbb{N}_0, 0 < \sigma < 1$, we denote by $W^{s,p}(D)$ the Sobolev space with norm

$$\|v\|_{W^{s,p}(D)} := \Big(\|v\|^p_{W^{m,p}(D)} + \sum_{|\alpha| = m} \int\limits_\Omega \int\limits_\Omega \frac{|D^\alpha v(\mathbf{x}) - D^\alpha v(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^{d+\sigma p}} \, d\mathbf{x} \, d\mathbf{y} \Big)^{1/p}.$$

We refer to $W_0^{s,p}(D)$ as the closure of $C_0^\infty(D)$ in $W_\omega^{s,p}(D)$. For $s < 0$, we denote by $W^{-s,p}(D)$ the dual space of $W_0^{-s,q}(D), p^{-1} + q^{-1} = 1$. In case $p = 2$, the spaces $W^{s,2}(D)$ are Hilbert spaces. We will write $H^s(D)$ instead of $W^{s,2}(D)$ and refer to $(\cdot,\cdot)_{H^s(D)}$ and $\|\cdot\|_{H^s(D)}$ as the inner products and associated norms. In the sequel, for two quantities A and B we will use the notation $A \lesssim B$, if there exists a positive constant $C > 0$ only depending on the data of the problem such that $A \le CB$.

2.2 Elliptic optimal control problem with distributed controls

We assume $\Omega \subset \mathbb{R}^2$ to be a bounded polygonal domain with boundary $\Gamma = \partial\Omega$ and denote by A the linear second order elliptic differential operator

$$Ay \;:=\; -\nabla \cdot a\nabla y \;+\; cy \;, \tag{2.1}$$

where $a = a(x), x \in \Omega$, is a symmetric, uniformly positive definite matrix-valued function and $c = c(x), x \in \Omega$, stands for a scalar nonnegative function. Given a desired state $y^d \in L^2(\Omega)$, a shift control $u^d \in L^2(\Omega)$, and a regularization parameter $\alpha > 0$ as well as a forcing term $f \in L^2(\Omega)$, we consider the following elliptic optimal control problem:
Find $(y, u) \in V \times W$, where $V := H_0^1(\Omega)$ and $W := L^2(\Omega)$, such that

$$\inf_{y,u} J(y,u), \tag{2.2a}$$

$$J(y,u) := \frac{1}{2}\|y - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2}\|u - u^d\|_{0,\Omega}^2, \tag{2.2b}$$

subject to

$$Ay = f + u \quad \text{in } \Omega, \tag{2.2c}$$

$$y = 0 \quad \text{on } \Gamma. \tag{2.2d}$$

The existence and uniqueness of an optimal solution can be easily shown (cf., e.g., [25, 50, 60]).

Theorem 2.1 *Under the above assumptions on the data of the problem, the distributed elliptic optimal control problem* (2.2a)-(2.2d) *admits a unique solution* $(y, u) \in V \times W$.

Proof. Introducing $G : W \to V$ as the control-to-state operator which assigns to a control $u \in W$ the solution $y = G(u)$ of the state equation (2.2c),(2.2d), the control reduced form of the optimal control problem (2.2a)-(2.2d) reads as

follows:
Find $u \in W$ such that

$$\inf_u J_{red}(u), \quad J_{red}(u) := J(G(u), u). \tag{2.3}$$

Let $(u_n)_{\mathbb{N}}, u_n \in W, n \in \mathbb{N}$, be a minimizing sequence, i.e., $J_{red}(u_n) \to \inf_u J_{red}(u)$ as $n \to \infty$. Due to the boundedness of $(u_n)_{\mathbb{N}}$ there exist a subsequence $\mathbb{N}' \subset \mathbb{N}$ and $u^* \in W$ such that $u_n \rightharpoonup u^*$ in W as $\mathbb{N}' \ni n \to \infty$. Since the objective functional J_{red} is lower semi-continuous and convex, it is weakly lower semi-continuous and hence, we have

$$\text{w - lim inf}_{n \in \mathbb{N}'} \; J_{red}(u_n) \geq J_{red}(u^*),$$

which shows that u^* solves (2.3). The uniqueness follows readily from the strict convexity of J_{red}. $\square$

2.3 Optimality conditions

Due to the convexity of the objective functional (2.2b) the first order necessary optimality conditions are sufficient as well.

Theorem 2.2 *Let* $(y, u) \in V \times W$ *be the unique solution of* (2.2a)-(2.2d). *Then, there exists an adjoint state* $p \in V$ *such that the triple* $(y, u, p) \in V \times W \times V$ *satisfies the optimality system*

$$Ay = f + u \quad \text{in } \Omega, \tag{2.4a}$$

$$y = 0 \quad \text{on } \Gamma, \tag{2.4b}$$

$$Ap = y^d - y \quad \text{in } \Omega, \tag{2.4c}$$

$$p = 0 \quad \text{on } \Gamma, \tag{2.4d}$$

$$p = \alpha(u - u^d) \quad \text{in } \Omega. \tag{2.4e}$$

Proof. Denoting by $J'_{red}(u) \in W^*$ the Gâteaux derivative of J_{red} in the optimal control $u \in W$, the necessary optimality condition for (2.3) reads

$$\begin{aligned} \langle J'_{red}(u), w \rangle_{W^*,W} &= (G(u) - y^d, G(w))_{0,\Omega} + \alpha(u - u^d, w)_{0,\Omega} \\ &= (G^*(G(u) - y^d) + \alpha(u - u^d), w)_{0,\Omega} = 0, \quad w \in W. \end{aligned} \tag{2.5}$$

Setting $p = -G^*(G(u) - y^d)$ and observing $y = G(u)$ as well as $G^* = G$, the optimality condition (2.5) implies that $p \in V$ satisfies (2.4c)-(2.4e), whereas $y \in V$ satisfies (2.4a),(2.4b) by definition of G. $\square$

If we substitute u in (2.4a) by means of (2.4e) according to $u = \alpha^{-1}p + u^d$, the operator-theoretic form of the optimality system can be stated as

$$\begin{pmatrix} A & -\alpha^{-1}I \\ I & A \end{pmatrix} \begin{pmatrix} y \\ p \end{pmatrix} = \begin{pmatrix} f + u^d \\ y^d \end{pmatrix}. \tag{2.6}$$

Chapter 3

Primal mixed formulation of the optimality system

Both the state equation (2.4a) and the adjoint state equation (2.4c) are linear second order elliptic equations that can be formally written as first order systems. In particular, introducing the fluxes

$$\boldsymbol{\lambda}_y := a\nabla y, \quad \boldsymbol{\lambda}_p := a\nabla p, \tag{3.1}$$

the optimality system (2.6) reads

$$\begin{pmatrix} a^{-1}I & -\nabla & 0 & 0 \\ -\nabla\cdot & cI & 0 & -\alpha^{-1}I \\ 0 & 0 & a^{-1}I & -\nabla \\ 0 & I & -\nabla\cdot & cI \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda}_y \\ y \\ \boldsymbol{\lambda}_p \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ f + u^d \\ 0 \\ y^d \end{pmatrix}. \tag{3.2}$$

We refer to (3.2) as the primal mixed formulation of the optimality system (2.6). Setting $\mathbf{Q} := \mathbf{L}^2(\Omega)^2$, its weak form amounts to the computation of $(\boldsymbol{\lambda}_y, y, \boldsymbol{\lambda}_p, p) \in \mathbf{Q} \times V \times \mathbf{Q} \times V$ such that for all $\mathbf{q} \in \mathbf{Q}$ and $v \in V$ the following system of variational equations holds true:

$$a_P(\boldsymbol{\lambda}_y, \mathbf{q}) - b_P(\mathbf{q}, y) = \ell_1(\mathbf{q}), \tag{3.3a}$$

$$b_P(\boldsymbol{\lambda}_y, v) + c_P(y, v) - \alpha^{-1} d_P(p, v) = \ell_2(v), \tag{3.3b}$$

$$a_P(\boldsymbol{\lambda}_p, \mathbf{q}) - b_P(\mathbf{q}, p) = \ell_3(\mathbf{q}), \tag{3.3c}$$

$$b_P(\boldsymbol{\lambda}_p, v) + c_P(p, v) + d_P(y, v) = \ell_4(v). \tag{3.3d}$$

Here, the bilinear forms $a_P(\cdot,\cdot) : \mathbf{Q} \times \mathbf{Q} \to \mathbb{R}, b_P(\cdot,\cdot) : \mathbf{Q} \times V \to \mathbb{R}, c_P(\cdot,\cdot) : V \times V \to \mathbb{R}$, and $d_P(\cdot,\cdot) : V \times V \to \mathbb{R}$ are given by

$$a_P(\mathbf{p},\mathbf{q}) := \int_\Omega a^{-1}\mathbf{p}\cdot\mathbf{q}\; d\mathbf{x}, \quad \mathbf{p},\mathbf{q} \in \mathbf{Q}, \tag{3.4a}$$

$$b_P(\mathbf{p},v) := \int_\Omega \mathbf{p}\cdot\nabla v\; d\mathbf{x}, \quad \mathbf{p} \in \mathbf{Q},\ v \in V, \tag{3.4b}$$

$$c_P(v,w) := \int_\Omega cvw\; d\mathbf{x}, \quad v,w \in V, \tag{3.4c}$$

$$d_P(v,w) := \int_\Omega vw\; d\mathbf{x}, \quad v,w \in V, \tag{3.4d}$$

whereas the functionals $\ell_{2\nu-1} : \mathbf{Q} \to \mathbb{R}, \ell_{2\nu} : V \to \mathbb{R}, 1 \le \nu \le 2$, read as follows

$$\ell_{2\nu-1}(\mathbf{q}) := 0, \quad \mathbf{q} \in \mathbf{Q},\ 1 \le \nu \le 2, \tag{3.5a}$$

$$\ell_2(v) := \int_\Omega (f+u^d)v\; d\mathbf{x}, \quad v \in V, \tag{3.5b}$$

$$\ell_4(v) := \int_\Omega y^d v\; d\mathbf{x}, \quad v \in V. \tag{3.5c}$$

We denote by $A_P : \mathbf{Q} \to \mathbf{Q}^*, B_P : V \to \mathbf{Q}^*, C_P : V \to V^*$, and $D_P : V \to V^*$ the operators associated with the bilinear forms a_P, b_P, c_P, and d_P. Moreover, we set $z := (z_y, z_p)^T$, where $z_y := (\boldsymbol{\lambda}_y, y)^T, z_p := (\boldsymbol{\lambda}_p, p)^T$, and $\ell := (\ell_1,\ell_2,\ell_3,\ell_4)^T$. Then, the operator-theoretic form of the optimality system (3.3a)-(3.3d) is given by

$$\mathcal{K}z = \ell. \tag{3.6}$$

Here, $\mathcal{K}$ stands for the operator-valued 2×2 matrix

$$\mathcal{K} := \begin{pmatrix} \mathcal{L} & -\alpha^{-1}\mathcal{M} \\ \mathcal{M} & \mathcal{L} \end{pmatrix}, \tag{3.7}$$

where $\mathcal{L} : \mathbf{Q} \times V \to \mathbf{Q}^* \times V^*$ and $\mathcal{M} : \mathbf{Q} \times V \to \mathbf{Q}^* \times V^*$ denote the operators

$$\mathcal{L} := \begin{pmatrix} A_P & -B_P \\ B_P^* & C_P \end{pmatrix}, \quad \mathcal{M} := \begin{pmatrix} 0 & 0 \\ 0 & D_P \end{pmatrix}. \tag{3.8}$$

We will show that the operator $\mathcal{K} : (\mathbf{Q} \times V)^2 \to (\mathbf{Q}^* \times V^*)^2$ is a continuous linear operator which is bijective. Consequently, for any $\ell \in (\mathbf{Q}^* \times V^*)^2$ the optimality system (3.6) admits a unique solution z which continuously depends on the data. As a preliminary result, we will prove a similar statement for the operator $\mathcal{L}$.

Proposition 3.1 *The operator $\mathcal{L} : \mathbf{Q} \times V \to \mathbf{Q}^* \times V^*$ as given by* (3.8) *is a continuous linear and bijective operator. Hence, for any $\ell_1 \in \mathbf{Q}^*$ and $\ell_2 \in V^*$ the operator equation*

$$\mathcal{L}\begin{pmatrix} \boldsymbol{\lambda} \\ y \end{pmatrix} = \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix}$$

admits a unique solution $(\boldsymbol{\lambda}, y) \in \mathbf{Q} \times V$ and there holds

$$\|(\boldsymbol{\lambda}, y)\|_{\mathbf{Q}\times V} \lesssim \|(\ell_1, \ell_2)\|_{\mathbf{Q}^*\times V^*}. \tag{3.9}$$

Proof. The linearity and continuity of $\mathcal{L}$ are obvious. In order to prove bijectivity, in view of the fact that the coefficient functions a and c are uniformly positive definite and non-negative, respectively, for any $(\boldsymbol{\lambda}, y) \in \mathbf{Q} \times V$ we have (cf. [17])

$$\begin{aligned} &\frac{1}{5}\|(\boldsymbol{\lambda}, y)\|_{\mathbf{Q}\times V}\|(\boldsymbol{\lambda} - \nabla y, 2y)\|_{\mathbf{Q}\times V} \\ \le\ & \frac{1}{5}\Big(\|\boldsymbol{\lambda}\|_{\mathbf{Q}} + \|y\|_V\Big)\Big(\|\boldsymbol{\lambda}\|_{\mathbf{Q}} + 3\|y\|_V\Big) \\ \le\ & \|\boldsymbol{\lambda}\|^2_{\mathbf{Q}} + \|y\|^2_V \lesssim (\mathcal{L}(\boldsymbol{\lambda}, y))(\boldsymbol{\lambda} - \nabla y, 2y). \end{aligned} \tag{3.10}$$

This implies an inf-sup condition and hence, we deduce bijectivity by the generalized Lax-Milgram lemma (cf., e.g., [10, 12]). The estimate (3.9) is an immediate consequence of the fact that $\mathcal{L}^{-1}$ is a bounded linear operator. □

Theorem 3.2 *The operator $\mathcal{K} : (\mathbf{Q} \times V)^2 \to (\mathbf{Q}^* \times V^*)^2$ defined by* (3.7) *is a continuous linear and bijective operator. Consequently, for any $\ell \in (\mathbf{Q}^* \times V^*)^2$ the optimality system* (3.6) *has a unique solution $z \in (\mathbf{Q} \times V)^2$ and there holds*

$$\|z\|_{(\mathbf{Q}\times V)^2} \lesssim \|\ell\|_{(\mathbf{Q}^*\times V^*)^2}. \tag{3.11}$$

Proof. Evidently, the operator $\mathcal{K}$ is linear and continuous. For the proof of its bijectivity, we choose left and right transforms

$$\mathcal{K}_L := \begin{pmatrix} \alpha^{+1/2} I & 0 \\ 0 & -I \end{pmatrix}, \quad \mathcal{K}_R := \begin{pmatrix} \alpha^{-1/2} I & 0 \\ 0 & I \end{pmatrix},$$

where I stands for the identity in the respective function space. We consider the transformed operator

$$\tilde{\mathcal{K}} := \mathcal{K}_L \mathcal{K} \mathcal{K}_R := \begin{pmatrix} \mathcal{L} & -\alpha^{-1/2}\mathcal{M} \\ -\alpha^{-1/2}\mathcal{M} & -\mathcal{L} \end{pmatrix}.$$

It suffices to verify bijectivity of $\tilde{\mathcal{K}}$. For any $z_y := (\boldsymbol{\lambda}_y, y)^T, z_p := (\boldsymbol{\lambda}_p, p)^T$, we choose $w_y := (\boldsymbol{\lambda}_y - \nabla y, 2y)^T$ and $w_p := -(\boldsymbol{\lambda}_p - \nabla p, 2p)^T$. It follows that

$$
\begin{aligned}
&(\tilde{\mathcal{K}}(z_y, z_p))(w_y, w_p) = \\
&(\mathcal{L}(\boldsymbol{\lambda}_y, y))(\boldsymbol{\lambda}_y - \nabla y, 2y) - \alpha^{-1/2}(\mathcal{M}(\boldsymbol{\lambda}_p, p))(\boldsymbol{\lambda}_y - \nabla y, 2y) \\
&+ \ \alpha^{-1/2}(\mathcal{M}(\boldsymbol{\lambda}_y, y))((\boldsymbol{\lambda}_p - \nabla p, 2p) + (\mathcal{L}(\boldsymbol{\lambda}_p, p))(\boldsymbol{\lambda}_p - \nabla p, 2p).
\end{aligned} \tag{3.12}
$$

Due to (3.8) we have

$$
\begin{aligned}
(\mathcal{M}(\boldsymbol{\lambda}_p, p))(\boldsymbol{\lambda}_y - \nabla y, 2y) &= 2\langle D_P p, y\rangle_{V^*,V}, \\
(\mathcal{M}(\boldsymbol{\lambda}_y, y))(\boldsymbol{\lambda}_p - \nabla p, 2p) &= 2\langle D_P y, p\rangle_{V^*,V}.
\end{aligned}
$$

Observing $\langle D_P p, y\rangle_{V^*,V} = \langle D_P y, p\rangle_{V^*,V}$ and (3.10), from (3.12) we deduce

$$
\begin{aligned}
&\frac{1}{5}\Big(\|(\boldsymbol{\lambda}_y, y)\|_{Q\times V} \|(\boldsymbol{\lambda}_y - \nabla y, 2y)\|_{Q\times V} \\
&+ \ \|(\boldsymbol{\lambda}_p, p)\|_{Q\times V} \|(\boldsymbol{\lambda}_p - \nabla p, 2p)\|_{Q\times V}\Big) \lesssim (\tilde{\mathcal{K}}(z_y, z_p))(w_y, w_p).
\end{aligned}
$$

As in the proof of Proposition 3.1 this implies bijectivity of $\tilde{\mathcal{K}}$. □

The previous theorem provides error estimates of approximate solutions of the optimality system (3.6) in terms of the associated residuals.

Corollary 3.3 *Let $\bar{z}_h = (\bar{z}_{\bar{y}_h}, \bar{z}_{\bar{p}_h})^T$ with $\bar{z}_{\bar{y}_h} = (\bar{\boldsymbol{\lambda}}_{\bar{y}_h}, \bar{y}_h)^T$ and $\bar{z}_{\bar{p}_h} = (\bar{\boldsymbol{\lambda}}_{\bar{p}_h}, \bar{p}_h)^T$ be an approximation of the solution $z = (z_y, z_p)^T$ of the optimality system* (3.6) *with $z_y = (\boldsymbol{\lambda}_y, y)^T$ and $z_p = (\boldsymbol{\lambda}_p, p)^T$. Then, there holds*

$$
\|z - \bar{z}_h\|_{(\mathbf{Q}\times V)^2} \lesssim \|Res\|_{(\mathbf{Q}^*\times V^*)^2}, \tag{3.13}
$$

where the residual Res is given by

$$
Res = (Res_1, Res_2, Res_3, Res_4)^T, \tag{3.14}
$$

and the residuals $Res_\nu, 1 \le \nu \le 4$, read as follows

$$
\begin{aligned}
&Res_1(\mathbf{q}) := \ell_1(\mathbf{q}) - a_P(\bar{\boldsymbol{\lambda}}_{\bar{y}_h}, \mathbf{q}) + b_P(\mathbf{q}, \bar{y}_h), \quad \mathbf{q} \in \mathbf{Q}, && \text{(3.15a)}\\
&Res_2(v) := \ell_2(v) - b_P(\bar{\boldsymbol{\lambda}}_{\bar{y}_h}, v) - c_P(\bar{y}_h, v) + \alpha^{-1} d_P(\bar{p}_h, v), \quad v \in V, && \text{(3.15b)}\\
&Res_3(\mathbf{q}) := \ell_3(\mathbf{q}) - a_P(\bar{\boldsymbol{\lambda}}_{\bar{p}_h}, \mathbf{q}) + b_P(\mathbf{q}, \bar{p}_h), \quad \mathbf{q} \in \mathbf{Q}, && \text{(3.15c)}\\
&Res_4(v) := \ell_4(v) - b_P(\bar{\boldsymbol{\lambda}}_{\bar{p}_h}, v) - c_P(\bar{p}_h, v) - d_P(\bar{y}_h, v), \quad v \in V. && \text{(3.15d)}
\end{aligned}
$$

Proof. The proof is an immediate consequence of Theorem 3.2. □

Chapter 4

Mixed finite element approximation of the optimality system

We consider a shape regular family $(\mathcal{T}_h(\Omega))_{h\in\mathbb{H}}$ of simplicial triangulations of the computational domain Ω where $\mathbb{H}$ is a null sequence of positive real numbers. We refer to $\mathcal{N}_h(D)$ as the set of vertices and to $\mathcal{E}_h(D)$ as the set of edges in $D \subseteq \Omega$. For $T \in \mathcal{T}_h(\Omega)$, we denote by h_T the diameter of T and set $h := \max\{h_T \mid T \in \mathcal{T}_h(Omega)\}$, and for $E \in \mathcal{E}_h(\Omega)$ we denote by h_E the length of the edge E. $P_k(D), k \in \mathbb{N}_0$, stands for the set of polynomials of degree $\leq k$ on D. From now on we will assume that the coefficient functions a and c in (2.1) are elementwise constant with respect to the triangulations $\mathcal{T}_h(\Omega), h \in \mathbb{H}$.
We refer to

$$V_h := \{v_h \in C_0(\Omega) \mid v_h|_T \in P_1(T), T \in \mathcal{T}_h(\Omega)\}$$

as the finite element space $V_h \subset V$ of P1 conforming finite elements and to

$$W_h := \{w_h \in L^2(\Omega) \mid w_h|_T \in P_0(T), T \in \mathcal{T}_h(\Omega)\}$$

as the linear space $W_h \subset W$ of elementwise constants with respect to the triangulation $\mathcal{T}_h(\Omega)$. We further denote by

$$\mathbf{Q}_h := \{\mathbf{q}_h \in \mathbf{H}(\mathrm{div};\Omega) \mid \mathbf{q}_h|_T \in \mathbf{RT}_0(T),\ T \in \mathcal{T}_h(\Omega)\}$$

the lowest order Raviart-Thomas space $\mathbf{RT}_0(\Omega;\mathcal{T}_h(\Omega))$ with respect to $\mathcal{T}_h(\Omega)$, where $\mathbf{RT}_0(T)$ stands for the lowest order Raviart-Thomas element

$$\mathbf{RT}_0(T) := \{\mathbf{q}_h(\mathbf{x}) = \mathbf{a} + b\mathbf{x},\ \mathbf{a} \in \mathbb{R}^2, b \in \mathbb{R},\ \mathbf{x} \in T\}.$$

We set

$$N_Q := \mathrm{card}(\mathcal{E}_h(\Omega)), \quad N_V := \mathrm{card}(\mathcal{N}_h(\Omega)), \quad N_W := \mathrm{card}(\mathcal{T}_h(\Omega)),$$

and denote by $\boldsymbol{\varphi}_i \in \mathbf{Q}_h, 1 \le i \le N_Q,\ \varphi_i \in V_h, 1 \le i \le N_V$, and $\psi_i \in W_h, 1 \le i \le N_W$, the canonical basis functions of $\mathbf{Q}_h, V_h$, and W_h, respectively, i.e.,

$$\mathbf{Q}_h = \mathrm{span}(\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_{N_Q}),\ V_h = \mathrm{span}(\varphi_1, \cdots, \varphi_{N_V}),\ W_h = \mathrm{span}(\psi_1, \cdots, \psi_{N_W}).$$

The mixed finite element approximation of the optimality system (2.6) is based on the primal-dual mixed formulation: Find $(\boldsymbol{\lambda}_y, y, \boldsymbol{\lambda}_p, p) \in (\mathbf{H}(\mathrm{div};\Omega) \times W)^2$ such that for all $\mathbf{q} \in \mathbf{H}(\mathrm{div};\Omega)$ and $w \in W$ there holds

$$a_D(\boldsymbol{\lambda}_y, \mathbf{q}) + b_D(\mathbf{q}, y) = \ell_1(\mathbf{q}), \tag{4.1a}$$

$$b_D(\boldsymbol{\lambda}_y, w) - c_D(y, w) + \alpha^{-1} d_D(p, w) = -\,\ell_2(w), \tag{4.1b}$$

$$a_D(\boldsymbol{\lambda}_p, \mathbf{q}) + b_D(\mathbf{q}, p) = \ell_3(\mathbf{q}), \tag{4.1c}$$

$$b_D(\boldsymbol{\lambda}_p, w) - c_D(p, w) - d_D(y, w) = -\,\ell_4(w). \tag{4.1d}$$

Here, the bilinear forms read

$$a_D(\mathbf{p}, \mathbf{q}) := \int_\Omega a^{-1} \mathbf{p} \cdot \mathbf{q}\ d\mathbf{x}, \quad \mathbf{p}, \mathbf{q} \in \mathbf{H}(\mathrm{div};\Omega), \tag{4.2a}$$

$$b_D(\mathbf{p}, w) := \int_\Omega \nabla \cdot \mathbf{p}\ w\ d\mathbf{x}, \quad \mathbf{p} \in \mathbf{H}(\mathrm{div};\Omega),\ w \in W, \tag{4.2b}$$

$$c_D(v, w) := \int_\Omega c v w\ d\mathbf{x}, \quad v, w \in W, \tag{4.2c}$$

$$d_D(v, w) := \int_\Omega v w\ d\mathbf{x}, \quad v, w \in W, \tag{4.2d}$$

whereas the functionals $\ell_{2\nu-1} : \mathbf{H}(\mathrm{div};\Omega) \to \mathbb{R}, \ell_{2\nu} : W \to \mathbb{R}, 1 \le \nu \le 2$, are given by

$$\ell_{2\nu-1}(\mathbf{q}) := 0, \quad \mathbf{q} \in \mathbf{H}(\mathrm{div};\Omega),\ 1 \le \nu \le 2, \tag{4.3a}$$

$$\ell_2(w) := \int_\Omega (f + u^d) w\ d\mathbf{x}, \quad w \in W, \tag{4.3b}$$

$$\ell_4(w) := \int_\Omega y^d w\ d\mathbf{x}, \quad w \in W. \tag{4.3c}$$

We denote by $f_h \in W_h$ the elementwise constant function with

$$f_h|_T := |T|^{-1} \int_T f\ d\mathbf{x}, \quad T \in \mathcal{T}_h(\Omega),$$

and define $y_h^d \in W_h$ and $u_h^d \in W_h$ analogously. Then, the mixed finite element approximation of the optimality system (2.6) amounts to the computation of

$(\boldsymbol{\lambda}_{y_h}, y_h, \boldsymbol{\lambda}_{p_h}, p_h) \in (\mathbf{Q}_h \times W_h)^2$ such that for all $\mathbf{q}_h \in \mathbf{Q}_h$ and $w_h \in W_h$ there holds

$$a_D(\boldsymbol{\lambda}_{y_h}, \mathbf{q}_h) + b_D(\mathbf{q}_h, y_h) = \ell_{h,1}(\mathbf{q}_h), \tag{4.4a}$$

$$b_D(\boldsymbol{\lambda}_{y_h}, w_h) - c_D(y_h, w_h) + \alpha^{-1} d_D(p_h, w_h) = -\ell_{h,2}(w_h), \tag{4.4b}$$

$$a_D(\boldsymbol{\lambda}_{p_h}, \mathbf{q}_h) + b_D(\mathbf{q}_h, p_h) = \ell_{h,3}(\mathbf{q}_h), \tag{4.4c}$$

$$b_D(\boldsymbol{\lambda}_{p_h}, w_h) - c_D(p_h, w_h) - d_D(y_h, w_h) = -\ell_{h,4}(w_h), \tag{4.4d}$$

where the functionals $\ell_{h,2\nu-1} : \mathbf{Q}_h \to \mathbb{R}, 1 \le \nu \le 2$, and $\ell_{h,2\nu} : W_h \to \mathbb{R}, 1 \le \nu \le 2$, are given by

$$\ell_{h,2\nu-1}(\mathbf{q}_h) := \ell_{2\nu-1}(\mathbf{q}_h), \quad \mathbf{q}_h \in \mathbf{Q}_h,\ 1 \le \nu \le 2, \tag{4.5a}$$

$$\ell_{h,2}(w_h) := \int_\Omega (f_h + u_h^d) w_h \, d\mathbf{x}, \quad w_h \in W_h, \tag{4.5b}$$

$$\ell_{h,4}(w_h) := \int_\Omega y_h^d w_h \, d\mathbf{x}, \quad w_h \in W_h. \tag{4.5c}$$

It follows readily from (4.4b) and (4.4d) that

$$(\nabla \cdot \boldsymbol{\lambda}_{y_h} - c y_h + \alpha^{-1} p_h + f_h + u_h^d)|_T = 0, \quad T \in \mathcal{T}_h(\Omega), \tag{4.6a}$$

$$(\nabla \cdot \boldsymbol{\lambda}_{p_h} - c p_h - y_h + y_h^d)|_T = 0, \quad T \in \mathcal{T}_h(\Omega). \tag{4.6b}$$

We denote by $\mathbf{A}_h \in \mathbb{R}^{N_Q \times N_Q}, \mathbf{B}_h \in \mathbb{R}^{N_Q \times N_W}, \mathbf{C}_h \in \mathbb{R}^{N_W \times N_W}$, and $\mathbf{D}_h \in \mathbb{R}^{N_W \times N_W}$ the matrices with entries

$$\begin{aligned}
(\mathbf{A}_h)_{i,j} &:= a_D(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j),\ 1 \le i,j \le N_Q,\\
(\mathbf{B}_h)_{i,j} &:= b_D(\boldsymbol{\varphi}_i, \psi_j),\ 1 \le i \le N_Q,\ 1 \le j \le N_W,\\
(\mathbf{C}_h)_{i,j} &:= c_D(\psi_i, \psi_j),\ 1 \le i,j \le N_W,\\
(\mathbf{D}_h)_{i,j} &:= d_D(\psi_i, \psi_j),\ 1 \le i,j \le N_W,
\end{aligned}$$

and we refer to $\mathbf{b}_h$ as the block vector $\mathbf{b}_h = (\mathbf{b}_{h,1}, \mathbf{b}_{h,2}, \mathbf{b}_{h,3}, \mathbf{b}_{h,4})^T$, where

$$\begin{aligned}
(\mathbf{b}_{h,2\nu-1})_i &:= \ell_{h,2\nu-1}(\boldsymbol{\varphi}_i),\ 1 \le i \le N_Q,\ 1 \le \nu \le 2,\\
(\mathbf{b}_{h,2\nu})_i &:= -\ell_{h,2\nu}(\psi_i),\ 1 \le i \le N_W,\ 1 \le \nu \le 2.
\end{aligned}$$

We further identify $\boldsymbol{\lambda}_{y_h}, \boldsymbol{\lambda}_{p_h}$ with vectors in $\mathbb{R}^{N_Q}$ and y_h, p_h with vectors in $\mathbb{R}^{N_W}$, and we set $\mathbf{z}_h = (\boldsymbol{\lambda}_{y_h}, y_h, \boldsymbol{\lambda}_{p_h}, p_h)^T$. Then, the mixed finite element approximation (4.4a)-(4.4d) represents a linear algebraic system of saddle point form

$$\mathbf{K}_h \mathbf{z}_h = \mathbf{b}_h. \tag{4.7}$$

The saddle point matrix $\mathbf{K}_h$ has the block structure

$$\mathbf{K}_h = \begin{pmatrix} \mathbf{L}_h & \alpha^{-1}\mathbf{M}_h \\ -\mathbf{M}_h & \mathbf{L}_h \end{pmatrix}, \tag{4.8}$$

where $\mathbf{L}_h$ and $\mathbf{M}_h$ are the 2×2 block matrices

$$\mathbf{L}_h := \begin{pmatrix} \mathbf{A}_h & \mathbf{B}_h \\ \mathbf{B}_h^T & -\mathbf{C}_h \end{pmatrix}, \quad \mathbf{M}_h := \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_h \end{pmatrix}. \tag{4.9}$$

Chapter 5

Numerical solution of the discretized optimality system

We will solve the linear algebraic system (4.7) by the preconditioned Richardson iteration [3]

$$\mathbf{z}_h^{(\nu+1)} = \mathbf{z}_h^{(\nu)} - \hat{\mathbf{K}}_h^{-1}\Big(\mathbf{K}_h \mathbf{z}_h^{(\nu)} - \mathbf{b}_h\Big), \quad \nu \in \mathbb{N}_0, \tag{5.1}$$

where $\hat{\mathbf{K}}_h$ is an appropriate preconditioner for $\mathbf{K}_h$ and $\mathbf{z}_h^{(0)}$ is a given initial iterate. The preconditioner $\hat{\mathbf{K}}_h$ will be constructed by means of left and right transforms.

5.1 Left and right transforms

Let $\mathbf{K}_{h,L}, \mathbf{K}_{h,R}$ be regular matrices. Then, (4.7) can be equivalently written as

$$\mathbf{K}_{h,L}\mathbf{K}_h\mathbf{K}_{h,R}\mathbf{K}_{h,R}^{-1}\mathbf{z}_h = \mathbf{K}_{h,L}\mathbf{b}_h. \tag{5.2}$$

Assuming $\tilde{\mathbf{K}}_h$ to be a suitable preconditioner for $\mathbf{K}_{h,L}\mathbf{K}_h\mathbf{K}_{h,R}$, we consider the transforming iteration

$$\mathbf{K}_{h,R}^{-1}\mathbf{z}_h^{(\nu+1)} = \mathbf{K}_{h,R}^{-1}\mathbf{z}_h^{(\nu)} - \tilde{\mathbf{K}}_h^{-1}\Big(\mathbf{K}_{h,L}\mathbf{K}_h\mathbf{z}_h^{(\nu)} - \mathbf{K}_{h,L}\mathbf{b}_h\Big). \tag{5.3}$$

Backtransformation yields

$$\begin{aligned} \mathbf{z}_h^{(\nu+1)} &= \mathbf{z}_h^{(\nu)} - \mathbf{K}_{h,R}\tilde{\mathbf{K}}_h^{-1}\mathbf{K}_{h,L}(\mathbf{K}_h\mathbf{z}_h^{(\nu)} - \mathbf{b}_h) && (5.4)\\ &= \mathbf{z}_h^{(\nu)} - (\mathbf{K}_{h,L}^{-1}\tilde{\mathbf{K}}_h\mathbf{K}_{h,R}^{-1})^{-1}(\mathbf{K}_h\mathbf{z}_h^{(\nu)} - \mathbf{b}_h). && (5.5)\end{aligned}$$

Consequently,

$$\hat{\mathbf{K}}_h := \mathbf{K}_{h,L}^{-1}\tilde{\mathbf{K}}_h\mathbf{K}_{h,R}^{-1} \tag{5.6}$$

is an appropriate preconditioner for $\mathbf{K}_h$.
We note that transforming iterations have been used as smoothers within multigrid methods [63] as well as for the iterative solution of KKT systems in PDE constrained optimization [42, 43, 44, 57, 58].

5.2 Construction of a preconditioner

As far as the construction of a preconditioner for $\mathbf{K}_h$ is concerned, we choose a left transform $\mathbf{K}_{h,L}$ and a right transform $\mathbf{K}_{h,R}$ as the following block-diagonal matrices

$$
\mathbf{K}_{h,L} = \begin{pmatrix} \alpha^{1/2}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix}, \quad \mathbf{K}_{h,R} = \begin{pmatrix} \alpha^{-1/2}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}. \tag{5.7}
$$

We thus obtain the symmetric block matrix

$$
\mathbf{K}_{h,L}\mathbf{K}_h\mathbf{K}_{h,R} = \begin{pmatrix} \mathbf{L}_h & \alpha^{-1/2}\mathbf{M}_h \\ \alpha^{-1/2}\mathbf{M}_h & -\mathbf{L}_h \end{pmatrix}.
$$

The Schur complement associated with $\mathbf{K}_{h,L}\mathbf{K}_h\mathbf{K}_{h,R}$ is given by

$$
\mathbf{S}_h = \mathbf{L}_h + \alpha^{-1}\mathbf{M}_h\mathbf{L}_h^{-1}\mathbf{M}_h.
$$

Consequently, we have

$$
\mathbf{K}_{h,L}\mathbf{K}_h\mathbf{K}_{h,R} = \begin{pmatrix} \mathbf{L}_h & \alpha^{-1/2}\mathbf{M}_h \\ \alpha^{-1/2}\mathbf{M}_h & -\mathbf{S}_h + \alpha^{-1}\mathbf{M}_h\mathbf{L}_h^{-1}\mathbf{M}_h \end{pmatrix}.
$$

With $\hat{\mathbf{L}}_h$ as a preconditioner for $\mathbf{L}_h$ and

$$
\hat{\mathbf{S}}_h := \tau^{-1} \ \mathrm{diag}(\hat{\mathbf{L}}_h + \alpha^{-1}\mathbf{M}_h\hat{\mathbf{L}}_h^{-1}\mathbf{M}_h), \ \tau > 0, \tag{5.8}
$$

as a symmetric Uzawa preconditioner for $\mathbf{K}_{h,L}\mathbf{K}_h\mathbf{K}_{h,R}$ we choose

$$
\tilde{\mathbf{K}}_h = \begin{pmatrix} \hat{\mathbf{L}}_h & \alpha^{-1/2}\mathbf{M}_h \\ \alpha^{-1/2}\mathbf{M}_h & -\hat{\mathbf{S}}_h + \alpha^{-1}\mathbf{M}_h\hat{\mathbf{L}}_h^{-1}\mathbf{M}_h \end{pmatrix}.
$$

Backtransformation yields

$$
\hat{\mathbf{K}}_h = (\mathbf{K}_{h,L})^{-1}\tilde{\mathbf{K}}_h(\mathbf{K}_{h,R})^{-1} = \begin{pmatrix} \hat{\mathbf{L}}_h & \alpha^{-1}\mathbf{M}_h \\ -\mathbf{M}_h & \hat{\mathbf{S}}_h - \alpha^{-1}\mathbf{M}_h\hat{\mathbf{L}}_h^{-1}\mathbf{M}_h \end{pmatrix}. \tag{5.9}
$$

We thus arrive at the following preconditioned Richardson iteration:

Algorithm (Preconditioned Richardson Iteration)

Step 1 (Initialization)
Choose an initial iterate $\mathbf{z}_h^{(0)}$, prescribe some tolerance $TOL > 0$, and set $\nu = 0$.

Step 2 (Iteration loop)
Step 2.1 (Computation of the residual)
Compute the residual with respect to $\mathbf{z}_h^{(\nu)}$:

$$\mathbf{d}_h^{(\nu)} = \mathbf{K}_h \mathbf{z}_h^{(\nu)} - \mathbf{b}_h.$$

Step 2.2 (Implementation of the preconditioner)
Solve the linear algebraic system

$$\begin{pmatrix} \hat{\mathbf{L}}_h & \alpha^{-1}\mathbf{M}_h \\ -\mathbf{M}_h & \hat{\mathbf{S}}_h - \alpha^{-1}\mathbf{M}_h \hat{\mathbf{L}}_h^{-1} \mathbf{M}_h \end{pmatrix} \mathbf{\Delta z}_h^{(\nu)} = \mathbf{d}_h^{(\nu)}. \tag{5.10}$$

Step 2.3 (Computation of the new iterate)
Compute

$$\mathbf{z}_h^{(\nu+1)} = \mathbf{z}_h^{(\nu)} + \mathbf{\Delta z}_h^{(\nu)}.$$

Step 2.4 (Termination criterion)
If

$$\frac{\|\mathbf{z}_h^{(\nu+1)} - \mathbf{z}_h^{(\nu)}\|}{\|\mathbf{z}_h^{(\nu+1)}\|} < TOL,$$

stop the algorithm. Otherwise, set $\nu := \nu + 1$ and go to Step 2.1

Consider the linear system (5.10) in the form

$$\begin{pmatrix} \hat{\mathbf{L}}_h & \alpha^{-1}\mathbf{M}_h \\ -\mathbf{M}_h & \hat{\mathbf{S}}_h - \alpha^{-1}\mathbf{M}_h \hat{\mathbf{L}}_h^{-1} \mathbf{M}_h \end{pmatrix} \begin{pmatrix} \mathbf{\Delta z}_{h,1}^{(\nu)} \\ \mathbf{\Delta z}_{h,2}^{(\nu)} \end{pmatrix} = \begin{pmatrix} \mathbf{d}_{h,1}^{(\nu)} \\ \mathbf{d}_{h,2}^{(\nu)} \end{pmatrix}.$$

Elimination of $\mathbf{\Delta z}_{h,1}^{(\nu)}$ results in the Schur complement system

$$\hat{\mathbf{S}}_h \mathbf{\Delta z}_{h,2}^{(\nu)} = \mathbf{d}_{h,2}^{(\nu)} + \mathbf{M}_h \hat{\mathbf{L}}_h^{-1} \mathbf{d}_{h,1}^{(\nu)}.$$

Hence, the solution of (5.10) can be reduced to the successive solution of the three linear systems

$$\begin{aligned} \hat{\mathbf{L}}_h \tilde{\mathbf{\Delta}}\mathbf{z}_{h,1}^{(\nu)} &= \mathbf{d}_{h,1}^{(\nu)} \\ \hat{\mathbf{S}}_h \mathbf{\Delta z}_{h,2}^{(\nu)} &= \mathbf{d}_{h,2}^{(\nu)} + \mathbf{M}_h \tilde{\mathbf{\Delta}}\mathbf{z}_{h,1}^{(\nu)}, \\ \hat{\mathbf{L}}_h \mathbf{\Delta z}_{h,1}^{(\nu)} &= \mathbf{d}_{h,1}^{(\nu)} - \alpha^{-1}\mathbf{M}_h \mathbf{\Delta z}_{h,2}^{(\nu)}. \end{aligned}$$

As far as an appropriate preconditioner $\hat{\mathbf{L}}_h$ for the saddle point matrix $\mathbf{L}_h$ (cf. (4.9)) is concerned, preconditioners for such matrices have been extensively studied in the literature. We refer to [14, 23, 45, 55].

Chapter 6

Residual-type a posteriori error estimation

This section deals with the derivation of a residual a posteriori error estimator and the proof of its reliability within the framework of a unified a posteriori error control [17]. As prerequisites we need some appropriate quasi-interpolation and reconstruction operators.

6.1 Quasi-interpolation and reconstruction operators

We first recall the definition of Clément's quasi-interpolation operator and state its stability and local approximation properties (cf., e.g., [61]).
For $a \in \mathcal{N}_h(\Omega)$ we denote by φ_a the nodal basis function with supporting point a, and we refer to D_a as the patch

$$D_a \ := \ \bigcup \{ \ T \in \mathcal{T}_h(\Omega) \ \mid \ a \in \mathcal{N}_h(T) \}.$$

We refer to π_a as the L^2-projection onto $P_1(D_a)$, i.e., $\pi_a(w), w \in W$ is given by

$$(\pi_a(w), z)_{L^2(D_a)} = (w, z)_{L^2(D_a)}, \quad z \in P_1(D_a).$$

Then, Clément's interpolation operator P_C is defined as follows

$$P_C w := \sum_{a \in \mathcal{N}_h(\Omega)} \pi_a(w) \ \varphi_a.$$

For $T \in \mathcal{T}_h(\Omega)$ and $E \in \mathcal{E}_h(\Omega)$ we denote by D_T and D_E the patches

$$D_T := \bigcup \ \{T' \in \mathcal{T}_h(\Omega) \ \mid \ \mathcal{N}_h(T') \cap \mathcal{N}_h(T) \ \neq \ \emptyset \ \},$$
$$D_E := \bigcup \ \{T' \in \mathcal{T}_h(\Omega) \ \mid \ \mathcal{N}_h(T') \cap \mathcal{N}_h(E) \ \neq \ \emptyset \}.$$

Then, for $v \in V$ and $T \in \mathcal{T}_h(\Omega), E \in \mathcal{E}_h(\Omega)$ there holds

$$\|P_C v\|_{0,T} \lesssim \|v\|_{0,D_T}, \tag{6.1a}$$
$$\|P_C v\|_{0,E} \lesssim \|v\|_{0,D_E}, \tag{6.1b}$$
$$\|\nabla P_C v\|_{0,T} \lesssim \|\nabla v\|_{0,D_T}, \tag{6.1c}$$
$$\|v - P_C v\|_{0,T} \lesssim h_T \ \|v\|_{1,D_T}, \tag{6.1d}$$
$$\|v - P_C v\|_{0,E} \lesssim h_E^{1/2} \ \|v\|_{1,D_E}. \tag{6.1e}$$

Further, due to the finite overlap of the patches D_T and D_E we have

$$\Big(\sum_{T \in \mathcal{T}_h(\Omega)} \|v\|^2_{\mu,D_T} \Big)^{1/2} \lesssim \|v\|_{\mu,\Omega}, \quad 0 \le \mu \le 1, \tag{6.2a}$$
$$\Big(\sum_{E \in \mathcal{E}_h(\Omega)} \|v\|^2_{\mu,D_E} \Big)^{1/2} \lesssim \|v\|_{\mu,\Omega}, \quad 0 \le \mu \le 1. \tag{6.2b}$$

The mixed finite element approximation (4.1a)-(4.1d) is a nonconforming approximation of the optimality system (3.3a)-(3.3d), since $y_h \in W_h \not\subset V$ and $p_h \in W_h \not\subset V$. In order to be able to apply Corollary 3.3, we need approximations $\bar{y}_h \in V$ of y_h and $\bar{p}_h \in V$ of p_h. These can be provided by a reconstruction operator

$$R : W_h \to V_h \subset V, \tag{6.3}$$

defined as follows

$$(Rw_h)(a) := N_a^{-1} \sum_{T \in \mathcal{T}_h(D_a)} w_h|_T, \quad w_h \in W_h, \tag{6.4}$$

where $D_a, a \in \mathcal{N}_h(\Omega)$, denotes the patch

$$D_a := \bigcup \{T \in \mathcal{T}_h(\Omega) \mid a \in \mathcal{N}_h(T)\},$$

and $N_a := \mathrm{card}(\mathcal{T}_h(D_a))$. As can be shown (cf., e.g., [18]), we have

$$\|Rw_h - w_h\|^2_W \lesssim \sum_{E \in \mathcal{E}_h(\Omega)} h_E \ \|[w_h]_E\|^2_{0,E}, \tag{6.5}$$

where $[w_h]_E, E = T_+ \cap T_-, T_\pm \in \mathcal{T}_h(\Omega)$, stands for the jump of $w_h \in W_h$ across E according to

$$[w_h]_E := w_h|_{T_+} - w_h|_{T_-}. \tag{6.6}$$

6.2 The residual a posteriori error estimator and its reliability

The residual a posteriori error estimator

(6.7)
$$\eta_h := \Big(\sum_{T\in\mathcal{T}_h(\Omega)} (\eta_T^2(\boldsymbol{\lambda}_{y_h})+\eta_T^2(\boldsymbol{\lambda}_{p_h}))+ \sum_{E\in\mathcal{E}_h(\Omega)} (\eta_E^2(\boldsymbol{\lambda}_{y_h})+\eta_E^2(\boldsymbol{\lambda}_{p_h})+\eta_E^2(y_h)+\eta_E^2(p_h))\Big)^{1/2}$$

consists of element residuals $\eta_T(\boldsymbol{\lambda}_{y_h}), \eta_T(\boldsymbol{\lambda}_{p_h})$, edge residuals $\eta_E(\boldsymbol{\lambda}_{y_h}), \eta_E(y_h)$ and $\eta_E(\boldsymbol{\lambda}_{p_h}), \eta_E(p_h)$. In particular, the element residuals residuals are given by

$$\eta_T(\boldsymbol{\lambda}_{y_h}) := h_T\ \|\text{curl } (a^{-1}\boldsymbol{\lambda}_{y_h})\|_{0,T}, \quad T\in\mathcal{T}_h(\Omega), \tag{6.8a}$$
$$\eta_T(\boldsymbol{\lambda}_{p_h}) := h_T\ \|\text{curl } (a^{-1}\boldsymbol{\lambda}_{p_h})\|_{0,T}, \quad T\in\mathcal{T}_h(\Omega). \tag{6.8b}$$

The edge residuals read as follows

$$\eta_E(\boldsymbol{\lambda}_{y_h}) := h_E^{1/2}\ \|[\mathbf{t}_E\cdot(a^{-1}\boldsymbol{\lambda}_{y_h})]_E\|_{0,E}, \quad E\in\mathcal{E}_h(\Omega), \tag{6.9a}$$
$$\eta_E(y_h) := h_E^{1/2}\ \|[y_h]_E\|_{0,E}, \quad E\in\mathcal{E}_h(\Omega), \tag{6.9b}$$
$$\eta_E(\boldsymbol{\lambda}_{p_h}) := h_E^{1/2}\ \|[\mathbf{t}_E\cdot(a^{-1}\boldsymbol{\lambda}_{p_h})]_E\|_{0,E}, \quad E\in\mathcal{E}_h(\Omega), \tag{6.9c}$$
$$\eta_E(p_h) := h_E^{1/2}\ \|[p_h]_E\|_{0,E}, \quad E\in\mathcal{E}_h(\Omega), \tag{6.9d}$$

where $\mathbf{t}_E$ stands for the tangential unit vector on $E\in\mathcal{E}_h(\Omega)$ and $[\mathbf{t}_E\cdot(a^{-1}\boldsymbol{\lambda}_{y_h})]_E$ and $[y_h]_E$ refer to the jumps of the tangential component of $a^{-1}\boldsymbol{\lambda}_{y_h}$ and of y_h across $E = T_+\cap T_-, T_\pm\in\mathcal{T}_h(\Omega)$ according to

$$\begin{aligned}
[\mathbf{t}_E\cdot(a^{-1}\boldsymbol{\lambda}_{y_h})]_E &:= (\mathbf{t}_E\cdot(a^{-1}\boldsymbol{\lambda}_{y_h}))|_{T_+} - (\mathbf{t}_E\cdot(a^{-1}\boldsymbol{\lambda}_{y_h}))|_{T_-},\\
[y_h]_E &:= y_h|_{T_+} - y_h|_{T_-}.
\end{aligned}$$

and (6.6). We note that $[\mathbf{t}_E\cdot(a^{-1}\boldsymbol{\lambda}_{p_h})]_E$ and $[p_h]_E$ are defined analogously. The a posteriori error analysis further involves data oscillations

$$osc_h := \Big(\sum_{T\in\mathcal{T}_h(\Omega)} (osc_T^2(f+u^d) + osc_T^2(y^d))\Big)^{1/2}, \tag{6.10}$$

where for data $g\in L^2(\Omega)$ the local term $osc_T(g)$ is given by

$$osc_T(g) := h_T\|g-g_T\|_{0,T}, \quad g_T := |T|^{-1}\int_T g\ d\mathbf{x}. \tag{6.11}$$

Theorem 6.1 *Let $z := (\boldsymbol{\lambda}_y, y, \boldsymbol{\lambda}_p, p)^T \in (\mathbf{Q}\times V)^2$ and $z_h := (\boldsymbol{\lambda}_{y_h}, y_h, \boldsymbol{\lambda}_{p_h}, p_h)^T \in (\mathbf{Q}_h \times W_h)^2$ be the solutions of the optimality system* (3.3a)-(3.3d) *and its mixed finite element approximation* (4.1a)-(4.1d)*. Further, let η_h and osc_h be the residual a posteriori error estimator and the data oscillations as given by* (6.7) *and* (6.10)*, respectively. Then there holds*

$$\|z - z_h\|_{(\mathbf{Q}\times W)^2} \lesssim \eta_h^2 + osc_h^2. \tag{6.12}$$

Proof. An application of Corollary 3.3 with $\bar{\boldsymbol{\lambda}}_{\bar{y}_h} = \boldsymbol{\lambda}_{y_h}, \bar{\boldsymbol{\lambda}}_{\bar{p}_h} = \boldsymbol{\lambda}_{p_h}$, and $\bar{y}_h = Ry_h, \bar{p}_h = Rp_h$, where $R : W_h \to V_h \subset V$ is the reconstruction operator (6.4), yields

$$\begin{aligned} \|z - z_h\|_{(\mathbf{Q}\times W)^2} &\lesssim \|z - \bar{z}_h\|^2_{(\mathbf{Q}\times V)^2} + \|\bar{z}_h - z_h\|^2_{(\mathbf{Q}\times W)^2} \\ &\lesssim \|\mathrm{Res}(\bar{z}_h)\|^2_{(\mathbf{Q}^*\times V^*)^2} + \|Ry_h - y_h\|^2_W + \|Rp_h - p_h\|^2_W. \end{aligned} \tag{6.13}$$

In particular, we have

$$\|\boldsymbol{\lambda}_y - \boldsymbol{\lambda}_{y_h}\|^2_{\mathbf{Q}} \lesssim \|\mathrm{Res}_1\|^2_{\mathbf{Q}^*}, \tag{6.14a}$$
$$\|y - \bar{y}_h\|^2_W \lesssim \|\mathrm{Res}_2\|^2_{\mathbf{Q}^*}, \tag{6.14b}$$
$$\|\boldsymbol{\lambda}_p - \boldsymbol{\lambda}_{p_h}\|^2_{\mathbf{Q}} \lesssim \|\mathrm{Res}_3\|^2_{\mathbf{Q}^*}, \tag{6.14c}$$
$$\|p - \bar{p}_h\|^2_W \lesssim \|\mathrm{Res}_4\|^2_{\mathbf{Q}^*}. \tag{6.14d}$$

As far as Res_1 is concerned, for $\mathbf{q} \in \mathbf{Q}$ there holds

$$\mathrm{Res}_1(\mathbf{q}) = \int_\Omega \Big(a^{-1}\boldsymbol{\lambda}_{y_h} - \nabla \bar{y}_h\Big)\ \mathbf{q}\ d\mathbf{x}. \tag{6.15}$$

By the Helmholtz decomposition (cf., e.g., [30]) there exists a function $\beta \in H^1(\Omega)$ such that

$$a^{-1}\boldsymbol{\lambda}_{y_h} = \nabla \bar{y}_h + \mathbf{curl}\ \beta, \tag{6.16a}$$
$$\|\mathbf{curl}\ \beta\|_{0,\Omega} = \inf_{v\in V} \|a^{-1}\boldsymbol{\lambda}_{y_h} - \nabla v\|_{0,\Omega}. \tag{6.16b}$$

Using (6.16a) in (6.15), it follows that

$$\|\mathrm{Res}_1\|_{\mathbf{Q}^*} \lesssim \|\mathbf{curl}\ \beta\|_{0,\Omega}. \tag{6.17}$$

Since $\mathbf{curl}\ \beta$ and $\nabla \bar{y}_h$ are orthogonal with respect to $(\cdot,\cdot)_{0,\Omega}$, we have

(6.18)
$$\|\mathbf{curl}\ \beta\|^2_{0,\Omega} = \int_\Omega \mathbf{curl}\ \beta \cdot \Big(\mathbf{curl}\ \beta + \nabla \bar{y}_h\Big)\ d\mathbf{x} = \int_\Omega \mathbf{curl}\ \beta \cdot a^{-1}\boldsymbol{\lambda}_{y_h}\ d\mathbf{x},$$

where we have used again the Helmholtz decomposition (6.16a). Now, for any $\beta_h \in V_h$ there holds $\mathbf{curl}\ \beta_h \in W_h^2$ which implies

$$\nabla \cdot \mathbf{curl}\ \beta_h|_T = 0, \quad T \in \mathcal{T}_h(\Omega). \tag{6.19}$$

Using $\mathbf{curl}\ \beta_h$ as a test function in (4.1a) and observing (6.19), we obtain

$$\begin{aligned}
&\int_\Omega \mathbf{curl}\ \beta \cdot a^{-1}\boldsymbol{\lambda}_{y_h}\ d\mathbf{x} = \sum_{T\in\mathcal{T}_h(\Omega)} \int_T \mathbf{curl}\ (\beta - \beta_h) \cdot a^{-1}\boldsymbol{\lambda}_{y_h}\ d\mathbf{x} \\
&= \sum_{T\in\mathcal{T}_h(\Omega)} \Big(- \int_T (\beta - \beta_h)\ \mathrm{curl}\ (a^{-1}\boldsymbol{\lambda}_{y_h})\ d\mathbf{x} + \int_{\partial T} (\beta - \beta_h) \cdot \mathbf{t}_{\partial T} \cdot (a^{-1}\boldsymbol{\lambda}_{y_h})\ ds \Big) \\
&= - \sum_{T\in\mathcal{T}_h(\Omega)} \int_T (\beta - \beta_h)\ \mathrm{curl}\ (a^{-1}\boldsymbol{\lambda}_{y_h})\ d\mathbf{x} + \sum_{E\in\mathcal{E}_h(\Omega)} (\beta - \beta_h)\ [\mathbf{t}_E \cdot (a^{-1}\boldsymbol{\lambda}_{y_h})]_E.
\end{aligned}$$

We choose $\beta_h = P_C\beta$ where P_C stands for Clément's quasi-interpolation operator. Then, straightforward estimation and (6.1d),(6.1e) as well as (6.2a),(6.2b) result in

$$\begin{aligned}
&|\int_\Omega \mathrm{curl}\ \beta \cdot a^{-1}\boldsymbol{\lambda}_{y_h}\ d\mathbf{x}| \le \\
&\sum_{T\in\mathcal{T}_h(\Omega)} h_T \|\mathrm{curl}(a^{-1}\boldsymbol{\lambda}_{y_h})\|_{0,T}\ h_T^{-1}\ \|\beta - P_C\beta\|_{0,T} \\
&\quad + \sum_{E\in\mathcal{E}_h(\Omega)} h_E^{1/2}\ \|[\mathbf{t}_E \cdot (a^{-1}\boldsymbol{\lambda}_{y_h})]_E\|_{0,E}\ \|h_E^{-1/2}\ \|\beta - P_C\beta\|_{0,E} \\
&\lesssim\ \|\beta\|_{1,\Omega}\Big((\sum_{T\in\mathcal{T}_h(\Omega)} h_T^2\ \|\mathbf{curl}(a^{-1}\boldsymbol{\lambda}_{y_h})\|_{0,T}^2)^{1/2} \\
&\qquad + (\sum_{E\in\mathcal{E}_h(\Omega)} h_E\ \|[\mathbf{t}_E \cdot (a^{-1}\boldsymbol{\lambda}_{y_h})]_E\|_{0,E}^2)^{1/2} \Big).
\end{aligned} \tag{6.20}$$

In view of (6.16b), we have $\|\beta\|_{1,\Omega} \lesssim \|\boldsymbol{\lambda}_y - \boldsymbol{\lambda}_{y_h}\|_{0,\Omega}$. Hence, (6.14a),(6.17),(6.18), and (6.20) imply

$$\|\boldsymbol{\lambda}_y - \boldsymbol{\lambda}_{y_h}\|_{0,\Omega}^2 \lesssim \sum_{T\in\mathcal{T}_h(\Omega)} \eta_T^2(\boldsymbol{\lambda}_{y_h}) + \sum_{E\in\mathcal{E}_h(\Omega)} \eta_E^2(\boldsymbol{\lambda}_{y_h}). \tag{6.21}$$

As far as Res_2 is concerned, observing (3.15b) and (4.6a), for $v \in V$ and

$v_h \in W_h$ with $v_h|_T = |T|^{-1} \int_T v \; d\mathbf{x}, T \in \mathcal{T}_h(\Omega)$, we find

$$\begin{aligned}
\mathrm{Res}_2(v) &= \int\limits_\Omega (f + u^d) v \; d\mathbf{x} - \int\limits_\Omega \boldsymbol{\lambda}_{y_h} \cdot \nabla v \; d\mathbf{x} - \int\limits_\Omega c\bar{y}_h \; d\mathbf{x} + \alpha^{-1} \int\limits_\Omega \bar{p}_h v \; d\mathbf{x} \\
&= \int\limits_\Omega (f_h + u_h^d) v \; d\mathbf{x} + \int\limits_\Omega \nabla \cdot \boldsymbol{\lambda}_{y_h} v \; d\mathbf{x} - \int\limits_\Omega c y_h \; d\mathbf{x} + \alpha^{-1} \int\limits_\Omega p_h v \; d\mathbf{x} \\
&+ \int\limits_\Omega (f - f_h + u^d - u_h^d) v \; d\mathbf{x} + \int\limits_\Omega c(y_h - \bar{y}_h) v \; d\mathbf{x} + \alpha^{-1} \int\limits_\Omega (p_h - \bar{p}_h) v \; d\mathbf{x} \\
&= \sum_{T \in \mathcal{T}_h(\Omega)} \Big(\int\limits_T (f - f_h + u^d - u_h^d)(v - v_h) \; d\mathbf{x} + \int\limits_T c(y_h - \bar{y}_h) v \; d\mathbf{x} \\
&\qquad + \; \alpha^{-1} \int\limits_T (p_h - \bar{p}_h) v \; d\mathbf{x} \Big).
\end{aligned}$$

Straightforward estimation yields

$$\begin{aligned}
|\mathrm{Res}_2(v)| \lesssim & \sum_{T \in \mathcal{T}_h(\Omega)} h_T \Big(\|f - f_h\|_{0,T} + \|u^d - u_h^d\|_{0,T} \Big) \; h_T^{-1} \; \|v - v_h\|_{0,T} \\
& + \sum_{T \in \mathcal{T}_h(\Omega)} \Big(\|y_h - \bar{y}_h\|_{0,T} + \|p_h - \bar{p}_h\|_{0,T} \Big) \; \|v\|_{0,T}.
\end{aligned} \tag{6.22}$$

Using the Poincaré inequality

$$\|v - v_h\|_{0,T} \lesssim h_T \; \|\nabla v\|_{0,T} \lesssim h_T \; \|v\|_{1,T},$$

and (6.5), from (6.14b) and (6.22) we deduce

$$\|y - y_h\|_{0,\Omega}^2 \lesssim \sum_{E \in \mathcal{E}_h(\Omega)} \Big(\eta_E^2(y_h) + \eta_E^2(p_h) \Big) + \sum_{T \in \mathcal{T}_h(\Omega)} osc_T^2(f + u^d). \tag{6.23}$$

The residuals Res_3 and Res_4 can be estimated from above in much the same way yielding

$$\|\boldsymbol{\lambda}_p - \boldsymbol{\lambda}_{p_h}\|_{0,\Omega}^2 \lesssim \sum_{T \in \mathcal{T}_h(\Omega)} \eta_T^2(\boldsymbol{\lambda}_{p_h}) + \sum_{E \in \mathcal{E}_h(\Omega)} \eta_E^2(\boldsymbol{\lambda}_{p_h}), \tag{6.24}$$

as well as

$$\|p - p_h\|_{0,\Omega}^2 \lesssim \sum_{E \in \mathcal{E}_h(\Omega)} \Big(\eta_E^2(y_h) + \eta_E^2(p_h) \Big) + \sum_{T \in \mathcal{T}_h(\Omega)} osc_T^2(y^d). \tag{6.25}$$

Finally, combining (6.21),(6.23),(6.24), and (6.25) gives the assertion. □

In the step MARK of the adaptive cycle we use Dörfler marking [22]. In particular, given a universal constant $0 < \Theta < 1$, we determine a set of elements $\mathcal{M}_T$ and a set of edges $\mathcal{M}_E$ such that

(6.26)
$$\begin{aligned}\Theta\ (\eta_h^2 + osc_h^2) \le & \sum_{T \in \mathcal{M}_T} \Big(\eta_T^2(\boldsymbol{\lambda}_{y_h}) + \eta_T^2(\boldsymbol{\lambda}_{p_h}) + osc_T^2(f + u^d) + osc_T^2(y^d) \Big) \\ & + \sum_{E \in \mathcal{M}_E} \Big(\eta_E^2(\boldsymbol{\lambda}_{y_h}) + \eta_E^2(y_h) + \eta_E^2(\boldsymbol{\lambda}_{p_h}) + \eta_E^2(p_h) \Big)\end{aligned}$$

The Dörfler marking can be realized by a greedy algorithm (cf., e.g., [36]).

Chapter 7

Numerical results

In this section, we provide a detailed documentation of numerical results for three examples illustrating the performance of the adaptive mixed finite element method (AMFEM).

In the following examples, we are interested in a comparison of the convergence rate between the AMFEM and uniform method, the discretization errors, residual-type a posteriori error estimators, and the local behavior of the a posteriori error estimators.

Let $\|z - z_h\|_{(\mathbf{Q}\times W)^2}$ denote the total error.

$$\|z - z_h\|_{(\mathbf{Q}\times W)^2} := \Big(\|\boldsymbol{\lambda}_y - \boldsymbol{\lambda}_{y_h}\|^2_{\mathbf{Q}} + \|y - y_h\|^2_W + \|\boldsymbol{\lambda}_p - \boldsymbol{\lambda}_{p_h}\|^2_{\mathbf{Q}} + \|p - p_h\|^2_W\Big)^{1/2}. \tag{7.1}$$

The element residuals

$$\eta^T_{h,1} := \Big(\sum_{T\in\mathcal{T}_h(\Omega)} \eta^2_T(\boldsymbol{\lambda}_{y_h})\Big)^{1/2}, \tag{7.2}$$

$$\eta^T_{h,2} := \Big(\sum_{T\in\mathcal{T}_h(\Omega)} \eta^2_T(\boldsymbol{\lambda}_{p_h})\Big)^{1/2}, \tag{7.3}$$

The edge residuals

$$\eta^E_{h,1} := \Big(\sum_{E\in\mathcal{E}_h(\Omega)} (\eta^2_E(\boldsymbol{\lambda}_{y_h}) + \eta^2_E(y_h))\Big)^{1/2}, \tag{7.4}$$

$$\eta^E_{h,2} := \Big(\sum_{E\in\mathcal{E}_h(\Omega)} (\eta^2_E(\boldsymbol{\lambda}_{p_h}) + \eta^2_E(p_h))\Big)^{1/2}, \tag{7.5}$$

and the data oscillations osc_h as given by (6.10).

Example 1: L-shaped domain.

We choose $\Omega = (-1, +1)^2 \backslash (0, +1) \times (-1, 0)$ and $a = c = 1$, as well as

$$y^d = (1 + 0.01) r^{2/3} \sin(\frac{2\varphi}{3}) \quad \text{(in polar coordinates)}$$
$$u^d = 0, f = 0.$$

The exact solution reads:

$$y = u = r^{2/3} \sin(\frac{2\varphi}{3}),$$
$$p = 0.01 r^{2/3} \sin(\frac{2\varphi}{3}).$$

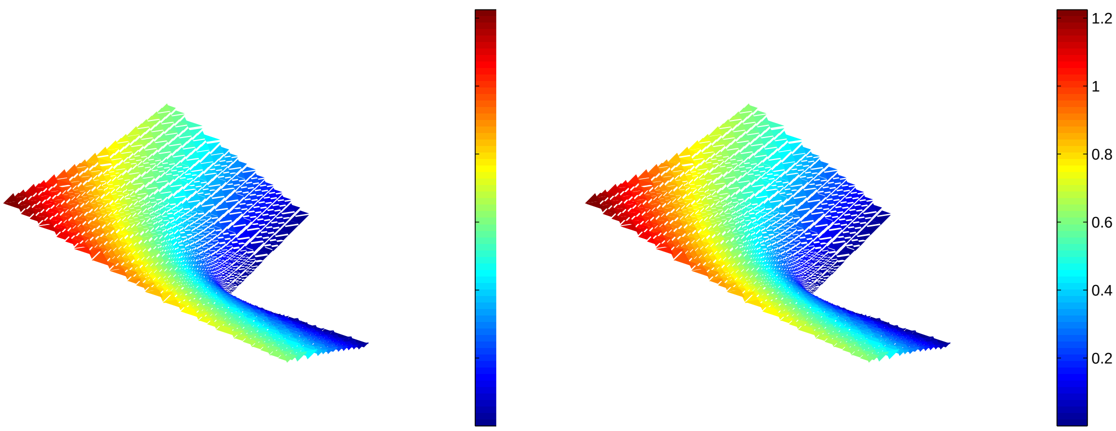

Figure 7.1: Example 1: Generated state y (left) and control u (right) after 20 cycles of the adaptive algorithm

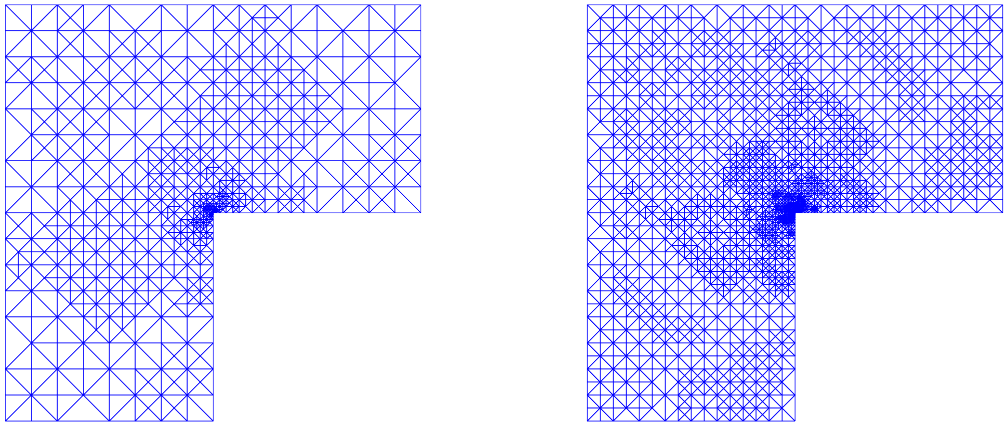

Figure 7.2: Example 1: Adaptively refined triangulations after 15 cycles(left) and 20 cycles(right) of the adaptive algorithm

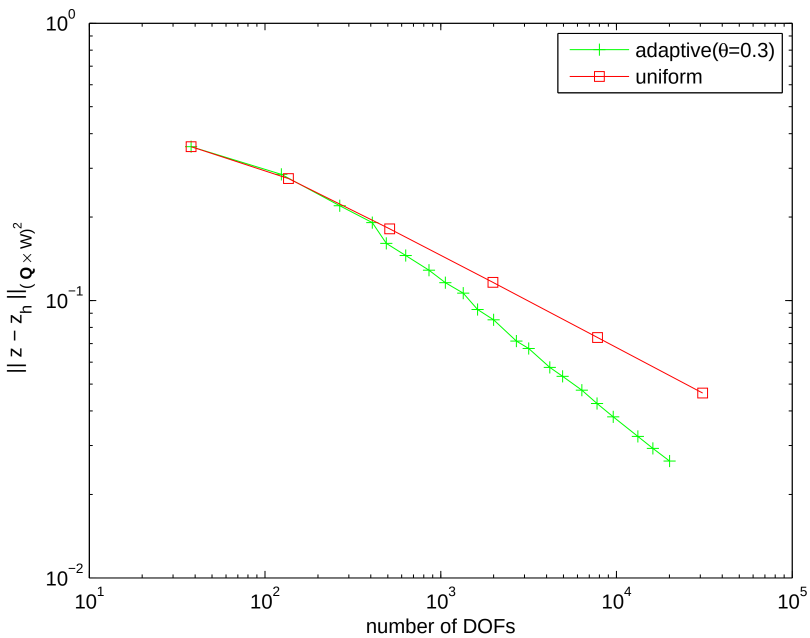

Figure 7.3: Example 1: Adaptive versus uniform refinement for the total error

Figure 7.3 provides a comparison between adaptive and uniform refinement. On a logarithmic scale, the decrease in the total error $\|z - z_h\|_{(\mathbf{Q}\times W)^2}$ is shown as a function of the degrees of freedom (DOF).

Table 7.1: Example 1: Convergence history of the AMFEM, Part I: Discretization errors for the flux of the state, the state, the control, the flux of the adjoint state, and the adjoint state

ℓ	$\mathbf{N_{DOF}}$	$\|\boldsymbol{\lambda}_\mathbf{y} - \boldsymbol{\lambda}_{\mathbf{y_h}}\|_{0,\Omega}$	$\|\mathbf{y} - \mathbf{y_h}\|_{0,\Omega}$	$\|\mathbf{u} - \mathbf{u_h}\|_{0,\Omega}$	$\|\boldsymbol{\lambda}_\mathbf{p} - \boldsymbol{\lambda}_{\mathbf{p_h}}\|_{0,\Omega}$	$\|\mathbf{p} - \mathbf{p_h}\|_{0,\Omega}$
0	**38**	**2.57e-01**	**2.50e-01**	**4.08e-01**	**8.21e-03**	**4.08e-03**
1	**124**	**2.32e-01**	**1.66e-01**	**2.05e-01**	**3.85e-03**	**2.05e-03**
2	**266**	**1.82e-01**	**1.23e-01**	**1.29e-01**	**2.00e-03**	**1.29e-03**
3	**408**	**1.51e-01**	**1.17e-01**	**1.18e-01**	**1.46e-03**	**1.18e-03**
4	**490**	**1.33e-01**	**9.05e-02**	**9.35e-02**	**1.42e-03**	**9.35e-04**
5	**632**	**1.14e-01**	**8.97e-02**	**9.09e-02**	**1.19e-03**	**9.09e-04**
6	**858**	**1.02e-01**	**7.85e-02**	**7.95e-02**	**1.07e-03**	**7.95e-04**
7	**1062**	**9.21e-02**	**7.04e-02**	**7.18e-02**	**9.98e-04**	**7.18e-04**
8	**1344**	**8.32e-02**	**6.61e-02**	**6.69e-02**	**8.73e-04**	**6.69e-04**
9	**1620**	**7.55e-02**	**5.41e-02**	**5.46e-02**	**7.82e-04**	**5.46e-04**
10	**2002**	**6.79e-02**	**5.14e-02**	**5.17e-02**	**6.98e-04**	**5.17e-04**
11	**2698**	**5.77e-02**	**4.22e-02**	**4.24e-02**	**5.87e-04**	**4.24e-04**
12	**3166**	**5.35e-02**	**4.06e-02**	**4.08e-02**	**5.44e-04**	**4.08e-04**
13	**4178**	**4.69e-02**	**3.32e-02**	**3.33e-02**	**4.76e-04**	**3.33e-04**
14	**4942**	**4.27e-02**	**3.19e-02**	**3.20e-02**	**4.31e-04**	**3.20e-04**
15	**6358**	**3.76e-02**	**2.91e-02**	**2.91e-02**	**3.78e-04**	**2.91e-04**
16	**7748**	**3.41e-02**	**2.54e-02**	**2.55e-02**	**3.43e-04**	**2.55e-04**
17	**9586**	**3.06e-02**	**2.27e-02**	**2.28e-02**	**3.07e-04**	**2.28e-04**
18	**13230**	**2.64e-02**	**1.88e-02**	**1.88e-02**	**2.65e-04**	**1.88e-04**
19	**16110**	**2.37e-02**	**1.73e-02**	**1.73e-02**	**2.37e-04**	**1.73e-04**
20	**20048**	**2.12e-02**	**1.57e-02**	**1.57e-02**	**2.12e-04**	**1.57e-04**

Table 7.2: Example 1: Convergence history of the AMFEM, Part II: Element and edge residuals, data oscillations

ℓ	$\mathbf{N_{DOF}}$	$\eta^{\mathbf{T}}_{\mathbf{h},\mathbf{1}}$	$\eta^{\mathbf{T}}_{\mathbf{h},\mathbf{2}}$	$\eta^{\mathbf{E}}_{\mathbf{h},\mathbf{1}}$	$\eta^{\mathbf{E}}_{\mathbf{h},\mathbf{2}}$	$\mathbf{osc_h}$
0	**38**	**0.00e+00**	**0.00e+00**	**1.15e+00**	**2.04e-02**	**8.18e-01**
1	**124**	**3.93e-17**	**0.00e+00**	**9.53e-01**	**8.95e-03**	**2.69e-01**
2	**266**	**5.72e-17**	**3.91e-19**	**7.57e-01**	**6.86e-03**	**1.82e-01**
3	**408**	**7.11e-17**	**8.82e-19**	**6.76e-01**	**6.43e-03**	**1.79e-01**
4	**490**	**6.69e-17**	**9.45e-19**	**5.79e-01**	**5.28e-03**	**9.56e-02**
5	**632**	**6.15e-17**	**6.55e-19**	**5.36e-01**	**5.09e-03**	**9.55e-02**
6	**858**	**7.35e-17**	**5.94e-19**	**4.82e-01**	**4.61e-03**	**5.56e-02**
7	**1062**	**7.46e-17**	**6.53e-19**	**4.31e-01**	**4.09e-03**	**5.30e-02**
8	**1344**	**7.20e-17**	**6.61e-19**	**3.97e-01**	**3.80e-03**	**4.94e-02**
9	**1620**	**8.04e-17**	**7.52e-19**	**3.58e-01**	**3.45e-03**	**3.19e-02**
10	**2002**	**7.30e-17**	**7.31e-19**	**3.28e-01**	**3.17e-03**	**2.96e-02**
11	**2698**	**6.94e-17**	**6.39e-19**	**2.75e-01**	**2.71e-03**	**1.99e-02**
12	**3166**	**6.24e-17**	**6.49e-19**	**2.59e-01**	**2.56e-03**	**1.80e-02**
13	**4178**	**6.22e-17**	**6.65e-19**	**2.26e-01**	**2.23e-03**	**1.15e-02**
14	**4942**	**6.65e-17**	**6.33e-19**	**2.09e-01**	**2.07e-03**	**1.09e-02**
15	**6358**	**7.48e-17**	**7.14e-19**	**1.85e-01**	**1.84e-03**	**9.45e-03**
16	**7748**	**7.95e-17**	**7.40e-19**	**1.67e-01**	**1.66e-03**	**7.10e-03**
17	**9586**	**7.25e-17**	**7.38e-19**	**1.50e-01**	**1.49e-03**	**5.79e-03**
18	**13230**	**6.42e-17**	**6.62e-19**	**1.29e-01**	**1.28e-03**	**3.78e-03**
19	**16110**	**7.17e-17**	**6.57e-19**	**1.16e-01**	**1.16e-03**	**3.28e-03**
20	**20048**	**7.47e-17**	**6.96e-19**	**1.05e-01**	**1.04e-03**	**2.69e-03**

Table 7.3: Example 1: Convergence history of the AMFEM, Part III: Average values of local a posteriori error estimators and data oscillations

ℓ	$\mathbf{N_{DOF}}$	$\hat{\eta}_{\mathbf{T}}(\boldsymbol{\lambda}_{\mathbf{y_h}})$	$\hat{\eta}_{\mathbf{T}}(\boldsymbol{\lambda}_{\mathbf{p_h}})$	$\hat{\eta}_{\mathbf{E}}(\boldsymbol{\lambda}_{\mathbf{y_h}})$	$\hat{\eta}_{\mathbf{E}}(\mathbf{y_h})$	$\hat{\eta}_{\mathbf{E}}(\boldsymbol{\lambda}_{\mathbf{p_h}})$	$\hat{\eta}_{\mathbf{E}}(\mathbf{p_h})$	$\mathbf{o\hat{s}c_T}$
0	**38**	**0.00e+00**	**0.00e+00**	**2.75e-01**	**2.05e-01**	**7.12e-03**	**2.56e-03**	**3.26e-01**
1	**124**	**2.52e-18**	**0.00e+00**	**1.18e-01**	**7.18e-02**	**1.05e-03**	**7.45e-04**	**5.34e-02**
2	**266**	**2.50e-18**	**1.52e-20**	**5.54e-02**	**3.36e-02**	**4.78e-04**	**3.41e-04**	**1.63e-02**
3	**408**	**2.94e-18**	**3.68e-20**	**3.87e-02**	**2.21e-02**	**3.56e-04**	**2.23e-04**	**9.70e-03**
4	**490**	**2.29e-18**	**3.46e-20**	**3.21e-02**	**1.88e-02**	**2.84e-04**	**1.90e-04**	**6.52e-03**
5	**632**	**1.65e-18**	**1.87e-20**	**2.61e-02**	**1.46e-02**	**2.39e-04**	**1.47e-04**	**4.95e-03**
6	**858**	**1.36e-18**	**1.24e-20**	**2.01e-02**	**1.09e-02**	**1.90e-04**	**1.10e-04**	**3.19e-03**
7	**1062**	**2.16e-18**	**1.72e-20**	**1.63e-02**	**8.91e-03**	**1.53e-04**	**8.95e-05**	**2.29e-03**
8	**1344**	**1.52e-18**	**1.53e-20**	**1.31e-02**	**7.06e-03**	**1.24e-04**	**7.09e-05**	**1.67e-03**
9	**1620**	**1.58e-18**	**1.63e-20**	**1.08e-02**	**5.85e-03**	**1.04e-04**	**5.87e-05**	**1.15e-03**
10	**2002**	**1.10e-18**	**1.34e-20**	**8.82e-03**	**4.76e-03**	**8.50e-05**	**4.77e-05**	**8.82e-04**
11	**2698**	**9.05e-19**	**9.09e-21**	**6.53e-03**	**3.57e-03**	**6.40e-05**	**3.58e-05**	**5.34e-04**
12	**3166**	**7.43e-19**	**8.39e-21**	**5.63e-03**	**3.05e-03**	**5.54e-05**	**3.05e-05**	**4.37e-04**
13	**4178**	**7.57e-19**	**8.68e-21**	**4.27e-03**	**2.31e-03**	**4.21e-05**	**2.32e-05**	**2.73e-04**
14	**4942**	**7.47e-19**	**7.71e-21**	**3.62e-03**	**1.96e-03**	**3.57e-05**	**1.96e-05**	**2.21e-04**
15	**6358**	**7.11e-19**	**7.13e-21**	**2.79e-03**	**1.54e-03**	**2.76e-05**	**1.54e-05**	**1.54e-04**
16	**7748**	**7.03e-19**	**7.07e-21**	**2.30e-03**	**1.25e-03**	**2.28e-05**	**1.25e-05**	**1.12e-04**
17	**9586**	**5.64e-19**	**6.08e-21**	**1.85e-03**	**1.02e-03**	**1.84e-05**	**1.02e-05**	**8.03e-05**
18	**13230**	**4.14e-19**	**4.46e-21**	**1.35e-03**	**7.38e-04**	**1.34e-05**	**7.39e-06**	**4.82e-05**
19	**16110**	**4.45e-19**	**4.00e-21**	**1.10e-03**	**6.04e-04**	**1.10e-05**	**6.04e-06**	**3.65e-05**
20	**20048**	**4.32e-19**	**4.12e-21**	**8.89e-04**	**4.85e-04**	**8.86e-05**	**4.85e-06**	**2.65e-05**

Example 2: Slit domain.

Let Ω be the hexagon with corners $(\pm 1, 0)$, $(\pm\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(\pm\frac{1}{2}, -\frac{\sqrt{3}}{2})$ and a slit along $y = 0$ and $x > 0$. The data of the prroblem are chosen according to $a = c = 1$ as well as

$$
\begin{aligned}
y^d =&(1 + 0.01)r^{1/4}\sin(\frac{\varphi}{4}) \quad \text{(in polar coordinates)}\\
u^d =&0, f = 0.
\end{aligned}
$$

The exact solution reads:

$$
\begin{aligned}
y = u = r^{1/4}\sin(\frac{\varphi}{4}),\\
p = 0.01r^{1/4}\sin(\frac{\varphi}{4}).
\end{aligned}
$$

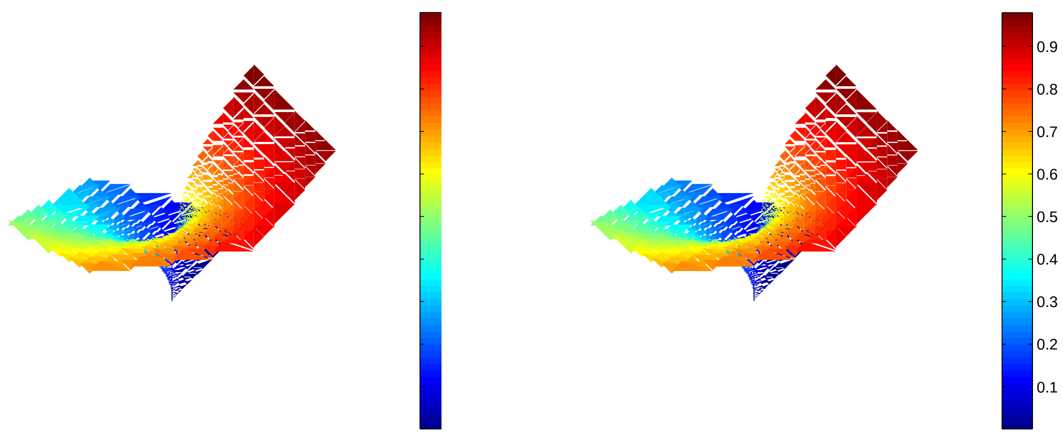

Figure 7.4: Example 2: Generated state y (left) and control u (right) after 20 cycles of the adaptive algorithm

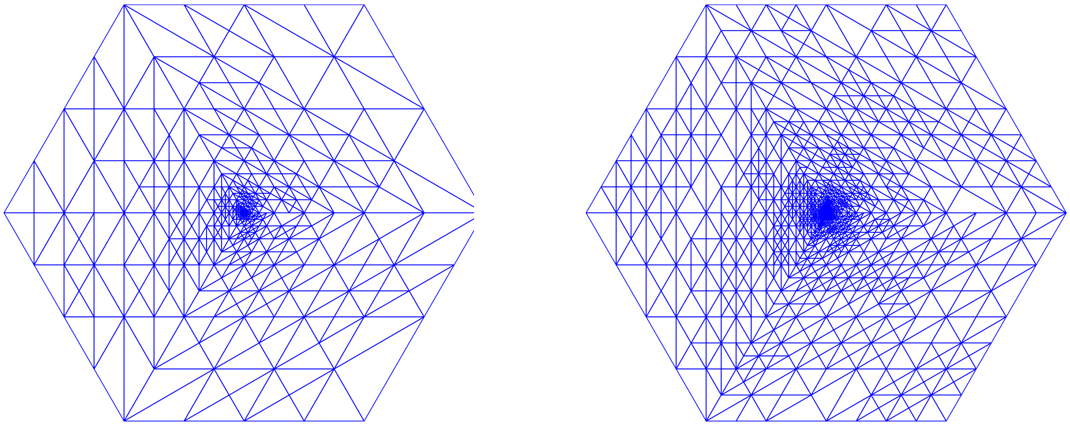

Figure 7.5: Example 2: Adaptively refined triangulations after 15 cycles(left) and 20 cycles(right) of the adaptive algorithm

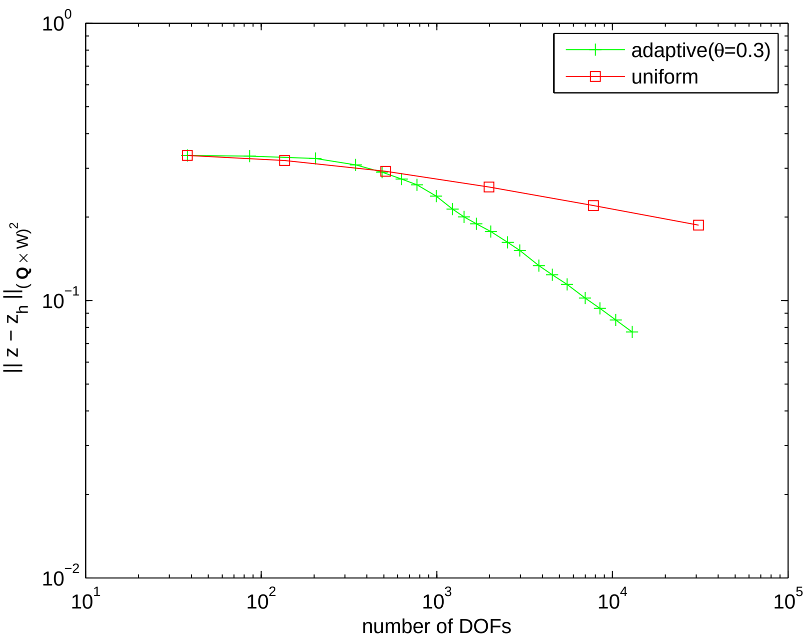

Figure 7.6: Example 2: Adaptive versus uniform refinement for the total error

Like in Example 1, Figure 7.6 provides a comparison between adaptive and uniform refinement. On a logarithmic scale, the decrease in the total error $\|z - z_h\|_{(\mathbf{Q}\times W)^2}$ is shown as a function of the degrees of freedom (DOF).

Table 7.4: Example 2: Convergence history of the AMFEM, Part I: Discretization errors for the flux of the state, the state, the control, the flux of the adjoint state, and the adjoint state

ℓ	$\mathbf{N_{DOF}}$	$\|\boldsymbol{\lambda}_\mathbf{y} - \boldsymbol{\lambda}_{\mathbf{y_h}}\|_{0,\Omega}$	$\|\mathbf{y} - \mathbf{y_h}\|_{0,\Omega}$	$\|\mathbf{u} - \mathbf{u_h}\|_{0,\Omega}$	$\|\boldsymbol{\lambda}_\mathbf{p} - \boldsymbol{\lambda}_{\mathbf{p_h}}\|_{0,\Omega}$	$\|\mathbf{p} - \mathbf{p_h}\|_{0,\Omega}$
0	**38**	**3.09e-01**	**1.27e-01**	**2.52e-01**	**7.03e-03**	**2.52e-03**
1	**86**	**3.15e-01**	**1.05e-01**	**1.90e-01**	**5.37e-03**	**1.90e-03**
2	**204**	**3.16e-01**	**7.85e-02**	**1.89e-01**	**6.36e-03**	**1.89e-03**
3	**346**	**3.03e-01**	**6.10e-02**	**1.26e-01**	**4.72e-03**	**1.26e-03**
4	**488**	**2.85e-01**	**5.58e-02**	**8.94e-02**	**3.60e-03**	**8.94e-04**
5	**630**	**2.69e-01**	**5.42e-02**	**7.20e-02**	**3.03e-03**	**7.20e-04**
6	**772**	**2.56e-01**	**5.38e-02**	**6.40e-02**	**2.72e-03**	**6.40e-04**
7	**992**	**2.33e-01**	**4.93e-02**	**5.60e-02**	**2.46e-03**	**5.60e-04**
8	**1230**	**2.10e-01**	**4.08e-02**	**4.41e-02**	**2.14e-03**	**4.41e-04**
9	**1428**	**1.96e-01**	**4.06e-02**	**4.28e-02**	**1.99e-03**	**4.28e-04**
10	**1680**	**1.85e-01**	**4.05e-02**	**4.18e-02**	**1.86e-03**	**4.18e-04**
11	**2032**	**1.73e-01**	**3.90e-02**	**4.03e-02**	**1.74e-03**	**4.03e-04**
12	**2534**	**1.59e-01**	**3.32e-02**	**3.41e-02**	**1.60e-03**	**3.41e-04**
13	**2972**	**1.49e-01**	**2.84e-02**	**2.92e-02**	**1.49e-03**	**2.92e-04**
14	**3812**	**1.31e-01**	**2.58e-02**	**2.61e-02**	**1.31e-03**	**2.61e-04**
15	**4538**	**1.21e-01**	**2.49e-02**	**2.52e-02**	**1.21e-03**	**2.52e-04**
16	**5512**	**1.12e-01**	**2.24e-02**	**2.29e-02**	**1.13e-03**	**2.29e-04**
17	**6986**	**1.00e-01**	**1.99e-02**	**2.01e-02**	**1.00e-03**	**2.01e-04**
18	**8486**	**9.17e-02**	**1.95e-02**	**1.96e-02**	**9.16e-04**	**1.96e-04**
19	**10444**	**8.35e-02**	**1.65e-02**	**1.65e-02**	**8.33e-04**	**1.65e-04**
20	**12962**	**7.58e-02**	**1.41e-02**	**1.41e-02**	**7.58e-04**	**1.41e-04**

Table 7.5: Example 2: Convergence history of the AMFEM, Part II: Element and edge residuals, data oscillations

ℓ	$\mathbf{N_{DOF}}$	$\eta^{\mathbf{T}}_{\mathbf{h,1}}$	$\eta^{\mathbf{T}}_{\mathbf{h,2}}$	$\eta^{\mathbf{E}}_{\mathbf{h,1}}$	$\eta^{\mathbf{E}}_{\mathbf{h,2}}$	$\mathbf{osc_h}$
0	**38**	**4.68e-17**	**1.01e-18**	**9.01e-01**	**9.70e-03**	**6.03e-01**
1	**86**	**5.60e-17**	**1.33e-18**	**1.05e+00**	**1.07e-02**	**5.12e-01**
2	**204**	**1.10e-16**	**9.48e-19**	**1.02e+00**	**1.26e-02**	**2.50e-01**
3	**346**	**1.35e-16**	**1.35e-18**	**9.98e-01**	**9.69e-03**	**8.94e-02**
4	**488**	**1.19e-16**	**1.32e-18**	**9.87e-01**	**8.88e-03**	**7.47e-02**
5	**630**	**1.27e-16**	**1.29e-18**	**9.81e-01**	**8.85e-03**	**7.40e-02**
6	**772**	**1.27e-16**	**1.19e-18**	**9.78e-01**	**8.94e-03**	**7.40e-02**
7	**992**	**1.21e-16**	**1.36e-18**	**9.00e-01**	**8.41e-03**	**5.81e-02**
8	**1230**	**1.29e-16**	**1.54e-18**	**8.26e-01**	**7.89e-03**	**3.61e-02**
9	**1428**	**1.46e-16**	**1.37e-18**	**7.85e-01**	**7.55e-03**	**3.61e-02**
10	**1680**	**1.37e-16**	**1.62e-18**	**7.51e-01**	**7.26e-03**	**3.60e-02**
11	**2032**	**1.45e-16**	**1.26e-18**	**7.12e-01**	**6.90e-03**	**3.40e-02**
12	**2534**	**1.47e-16**	**1.63e-18**	**6.58e-01**	**6.43e-03**	**2.61e-02**
13	**2972**	**1.57e-16**	**1.62e-18**	**6.20e-01**	**6.08e-03**	**2.01e-02**
14	**3812**	**1.48e-16**	**1.71e-18**	**5.49e-01**	**5.40e-03**	**1.76e-02**
15	**4538**	**1.67e-16**	**1.65e-18**	**5.10e-01**	**5.02e-03**	**1.65e-02**
16	**5512**	**1.76e-16**	**1.69e-18**	**4.72e-01**	**4.66e-03**	**1.31e-02**
17	**6986**	**1.69e-16**	**1.76e-18**	**4.23e-01**	**4.20e-03**	**1.02e-02**
18	**8486**	**1.72e-16**	**1.64e-18**	**3.88e-01**	**3.85e-03**	**1.01e-02**
19	**10444**	**1.62e-16**	**1.77e-18**	**3.53e-01**	**3.51e-03**	**6.59e-03**
20	**12962**	**1.68e-16**	**1.76e-18**	**3.22e-01**	**3.20e-03**	**4.89e-03**

Table 7.6: Example 2: Convergence history of the AMFEM, Part III: Average values of local a posteriori error estimators and data oscillations

ℓ	$\mathbf{N_{DOF}}$	$\hat{\eta}_{\mathbf{T}}(\boldsymbol{\lambda}_{\mathbf{y_h}})$	$\hat{\eta}_{\mathbf{T}}(\boldsymbol{\lambda}_{\mathbf{p_h}})$	$\hat{\eta}_{\mathbf{E}}(\boldsymbol{\lambda}_{\mathbf{y_h}})$	$\hat{\eta}_{\mathbf{E}}(\mathbf{y_h})$	$\hat{\eta}_{\mathbf{E}}(\boldsymbol{\lambda}_{\mathbf{p_h}})$	$\hat{\eta}_{\mathbf{E}}(\mathbf{p_h})$	$\mathbf{o\hat{s}c_T}$
0	38	1.26e-17	2.62e-19	3.15e-01	8.98e-02	3.04e-03	1.48e-03	2.29e-01
1	86	1.03e-17	1.62e-19	2.03e-01	4.83e-02	1.84e-03	6.92e-04	7.97e-02
2	204	9.48e-18	9.35e-20	1.02e-01	2.13e-02	1.10e-03	3.36e-04	1.73e-02
3	346	9.35e-18	9.63e-20	7.24e-02	1.34e-02	6.96e-04	1.72e-04	6.13e-03
4	488	7.27e-18	7.00e-20	6.00e-02	9.79e-03	5.50e-04	1.12e-04	3.73e-03
5	630	6.77e-18	7.04e-20	5.26e-02	7.72e-03	4.82e-04	8.37e-05	2.78e-03
6	772	5.90e-18	5.70e-20	4.72e-02	6.37e-03	4.38e-04	6.69e-05	2.25e-03
7	992	5.19e-18	5.61e-20	3.91e-02	4.92e-03	3.70e-04	5.13e-05	1.54e-03
8	1230	4.82e-18	5.77e-20	3.34e-02	4.03e-03	3.21e-04	4.15e-05	1.05e-03
9	1428	5.09e-18	4.57e-20	2.98e-02	3.48e-03	2.89e-04	3.56e-05	8.95e-04
10	1680	4.36e-18	4.93e-20	2.63e-02	2.98e-03	2.56e-04	3.03e-05	7.57e-04
11	2032	4.14e-18	3.80e-20	2.24e-02	2.45e-03	2.19e-04	2.49e-05	5.90e-04
12	2534	3.60e-18	3.95e-20	1.86e-02	2.03e-03	1.83e-04	2.06e-05	4.34e-04
13	2972	3.58e-18	3.81e-20	1.63e-02	1.73e-03	1.60e-04	1.75e-05	3.26e-04
14	3812	3.15e-18	3.44e-20	1.29e-02	1.37e-03	1.28e-04	1.38e-05	2.32e-04
15	4538	3.07e-18	3.01e-20	1.10e-02	1.16e-03	1.08e-04	1.17e-05	1.89e-04
16	5512	3.01e-18	2.84e-20	9.15e-03	9.49e-04	9.06e-05	9.54e-06	1.37e-04
17	6986	2.43e-18	2.63e-20	7.25e-03	7.50e-04	7.21e-05	7.52e-06	9.56e-05
18	8486	2.42e-18	2.37e-20	6.00e-03	6.21e-04	5.97e-05	6.23e-06	7.66e-05
19	10444	2.09e-18	2.18e-20	4.91e-03	5.02e-04	4.89e-05	5.03e-06	5.20e-05
20	12962	1.86e-18	1.96e-20	4.00e-03	4.06e-04	3.98e-05	4.07e-06	3.63e-05

Tables 7.4,7.5,7.6 document the convergence history of the AMFEM for Example 2 with the same legends as for Example 1.
Finally, Figure 7.6 displays the performance of the AMFEM in comparison to uniform refinement.

Example 3: Solution with a boundary layer.

We choose $\Omega = (0, +1)^2$ and $a = 1, c = 99$, as well as

$$
\begin{aligned}
y^d &= (1 + 0.01)(2cosh(10))^{-1}\Big(cosh(10x_1) + cosh(10x_2)\Big),\\
u^d &= 0, f = 0.
\end{aligned}
$$

The exact solution reads:

$$
\begin{aligned}
y =& \ (2cosh(10))^{-1}\Big(cosh(10x_1) + cosh(10x_2)\Big),\\
u =& \ -(2cosh(10))^{-1}\Big(cosh(10x_1) + cosh(10x_2)\Big),\\
p =& \ -0.01(2cosh(10))^{-1}\Big(cosh(10x_1) + cosh(10x_2)\Big).
\end{aligned}
$$

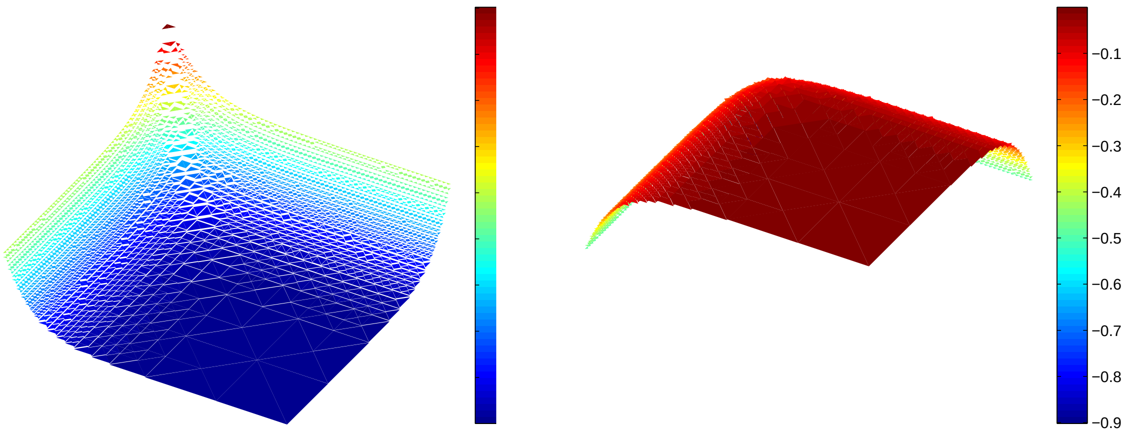

Figure 7.7: Example 2: Generated state y (left) and control u (right) after 20 cycles of the adaptive algorithm

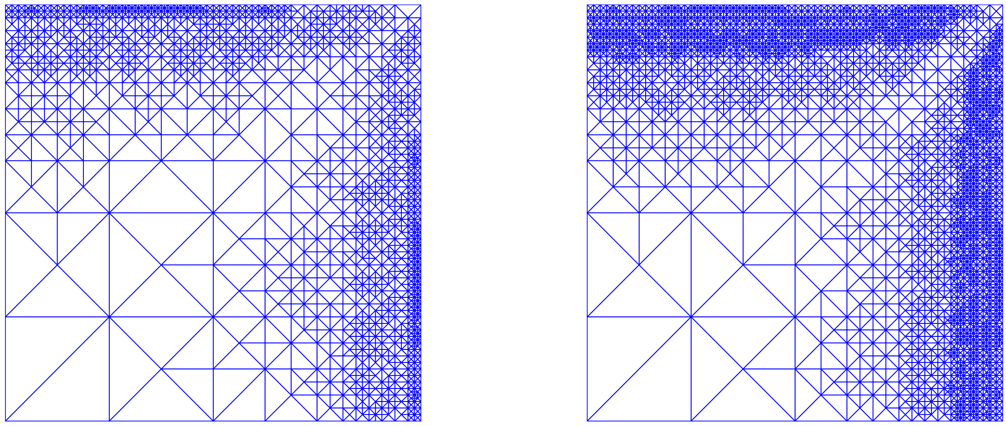

Figure 7.8: Example 2: Adaptively refined triangulations after 15 cycles(left) and 20(right) cycles of the adaptive algorithm

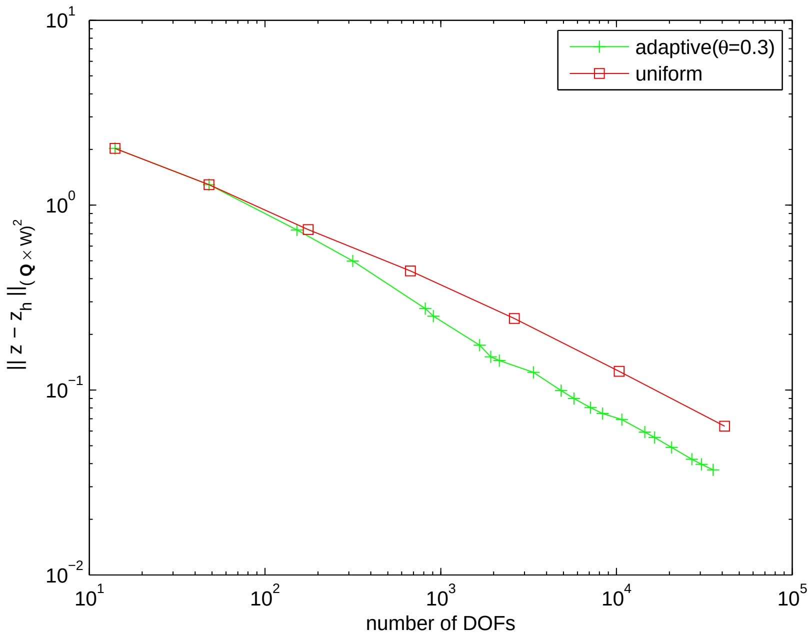

Figure 7.9: Example 2: Adaptive versus uniform refinement for the total error

Similarly, Figure 7.9 provides a comparison between adaptive and uniform refinement. On a logarithmic scale, the decrease in the total error $\|z - z_h\|_{(\mathbf{Q}\times W)^2}$ is shown as a function of the degrees of freedom (DOF).

Table 7.7: Example 3: Convergence history of the AMFEM, Part I: Discretization errors for the flux of the state, the state, the control, the flux of the adjoint state, and the adjoint state

ℓ	$\mathbf{N_{DOF}}$	$\|\boldsymbol{\lambda}_\mathbf{y} - \boldsymbol{\lambda}_{\mathbf{y_h}}\|_{0,\Omega}$	$\|\mathbf{y} - \mathbf{y_h}\|_{0,\Omega}$	$\|\mathbf{u} - \mathbf{u_h}\|_{0,\Omega}$	$\|\boldsymbol{\lambda}_\mathbf{p} - \boldsymbol{\lambda}_{\mathbf{p_h}}\|_{0,\Omega}$	$\|\mathbf{p} - \mathbf{p_h}\|_{0,\Omega}$
0	**14**	**2.01e+00**	**2.65e-01**	**2.54e-01**	**2.08e-02**	**2.54e-03**
1	**48**	**1.28e+00**	**1.76e-01**	**1.78e-01**	**1.28e-02**	**1.78e-03**
2	**152**	**7.28e-01**	**9.48e-02**	**9.80e-02**	**7.43e-03**	**9.80e-04**
3	**316**	**4.95e-01**	**6.85e-02**	**6.93e-02**	**5.07e-03**	**6.93e-04**
4	**816**	**2.74e-01**	**3.55e-02**	**3.59e-02**	**2.78e-03**	**3.59e-04**
5	**908**	**2.49e-01**	**2.85e-02**	**2.86e-02**	**2.52e-03**	**2.86e-04**
6	**1664**	**1.73e-01**	**2.20e-02**	**2.20e-02**	**1.75e-03**	**2.20e-04**
7	**1928**	**1.50e-01**	**1.96e-02**	**1.96e-02**	**1.51e-03**	**1.96e-04**
8	**2156**	**1.43e-01**	**1.68e-02**	**1.68e-02**	**1.44e-03**	**1.68e-04**
9	**3368**	**1.24e-01**	**1.49e-02**	**1.49e-02**	**1.24e-03**	**1.49e-04**
10	**4842**	**9.86e-02**	**1.23e-02**	**1.23e-02**	**9.87e-04**	**1.23e-04**
11	**5730**	**8.94e-02**	**1.10e-02**	**1.10e-02**	**8.95e-04**	**1.10e-04**
12	**7120**	**7.96e-02**	**1.01e-02**	**1.01e-02**	**7.97e-04**	**1.01e-04**
13	**8348**	**7.42e-02**	**8.80e-03**	**8.80e-03**	**7.43e-04**	**8.80e-05**
14	**10732**	**6.87e-02**	**8.12e-03**	**8.12e-03**	**6.88e-04**	**8.12e-05**
15	**14486**	**5.87e-02**	**7.28e-03**	**7.28e-03**	**5.88e-04**	**7.28e-05**
16	**16496**	**5.50e-02**	**6.93e-03**	**6.93e-03**	**5.50e-04**	**6.93e-05**
17	**20604**	**4.85e-02**	**6.24e-03**	**6.24e-03**	**4.86e-04**	**6.24e-05**
18	**26860**	**4.19e-02**	**5.20e-03**	**5.20e-03**	**4.19e-04**	**5.20e-05**
19	**30440**	**3.94e-02**	**4.86e-03**	**4.85e-03**	**3.94e-04**	**4.85e-05**
20	**35532**	**3.67e-02**	**4.59e-03**	**4.59e-03**	**3.67e-04**	**4.59e-05**

Table 7.8: Example 3: Convergence history of the AMFEM, Part II: Element and edge residuals, data oscillations

ℓ	$\mathbf{N_{DOF}}$	$\eta^{\mathbf{T}}_{\mathbf{h,1}}$	$\eta^{\mathbf{T}}_{\mathbf{h,2}}$	$\eta^{\mathbf{E}}_{\mathbf{h,1}}$	$\eta^{\mathbf{E}}_{\mathbf{h,2}}$	$\mathbf{osc_h}$
0	**14**	**0.00e+00**	**0.00e+00**	**6.46e-01**	**6.15e-02**	**9.02e-01**
1	**48**	**0.00e+00**	**0.00e+00**	**2.47e+00**	**2.54e-01**	**2.54e-01**
2	**152**	**1.57e-16**	**1.23e-18**	**2.59e+00**	**2.74e-02**	**9.59e-02**
3	**316**	**5.03e-17**	**1.47e-19**	**1.86e+00**	**1.92e-02**	**4.45e-02**
4	**816**	**1.02e-16**	**9.21e-19**	**1.13e+00**	**1.15e-02**	**1.64e-02**
5	**908**	**1.08e-16**	**7.48e-19**	**1.06e+00**	**1.08e-02**	**7.27e-03**
6	**1664**	**1.03e-16**	**1.02e-18**	**7.40e-01**	**7.46e-03**	**5.24e-03**
7	**1928**	**1.11e-16**	**1.09e-18**	**6.39e-01**	**6.41e-03**	**4.46e-03**
8	**2156**	**9.24e-17**	**9.69e-19**	**6.13e-01**	**6.14e-03**	**3.31e-03**
9	**3368**	**9.94e-17**	**1.08e-18**	**5.31e-01**	**5.31e-03**	**2.86e-03**
10	**4842**	**1.00e-16**	**1.01e-18**	**4.23e-01**	**4.24e-03**	**1.84e-03**
11	**5730**	**9.43e-17**	**9.25e-19**	**3.84e-01**	**3.84e-03**	**1.36e-03**
12	**7120**	**9.94e-17**	**1.07e-18**	**3.41e-01**	**3.41e-03**	**1.23e-03**
13	**8348**	**1.02e-16**	**1.02e-18**	**3.16e-01**	**3.16e-03**	**9.83e-04**
14	**10732**	**9.52e-17**	**1.03e-18**	**2.92e-01**	**2.92e-03**	**8.01e-04**
15	**14486**	**9.75e-17**	**1.10e-18**	**2.50e-01**	**2.51e-03**	**6.75e-04**
16	**16496**	**9.51e-17**	**1.14e-18**	**2.35e-01**	**2.35e-03**	**5.99e-04**
17	**20604**	**1.00e-16**	**1.02e-18**	**2.06e-01**	**2.07e-03**	**5.36e-04**
18	**26860**	**1.08e-16**	**9.83e-19**	**1.79e-03**	**1.79e-03**	**3.76e-04**
19	**30440**	**1.03e-16**	**9.97e-19**	**1.68e-01**	**1.69e-03**	**3.10e-04**
20	**35532**	**9.86e-17**	**1.01e-18**	**1.57e-01**	**1.57e-03**	**2.85e-04**

Table 7.9: Example 3: Convergence history of the AMFEM, Part III: Average values of local a posteriori error estimators and data oscillations

ℓ	$\mathbf{N_{DOF}}$	$\hat{\eta}_{\mathbf{T}}(\boldsymbol{\lambda}_{\mathbf{y_h}})$	$\hat{\eta}_{\mathbf{T}}(\boldsymbol{\lambda}_{\mathbf{p_h}})$	$\hat{\eta}_{\mathbf{E}}(\boldsymbol{\lambda}_{\mathbf{y_h}})$	$\hat{\eta}_{\mathbf{E}}(\mathbf{y_h})$	$\hat{\eta}_{\mathbf{E}}(\boldsymbol{\lambda}_{\mathbf{p_h}})$	$\hat{\eta}_{\mathbf{E}}(\mathbf{p_h})$	$\mathbf{o\hat{s}c_T}$
0	**14**	**0.00e+00**	**0.00e+00**	**1.57e-16**	**2.00e-01**	**1.96e-18**	**2.00e-03**	**6.38e-01**
1	**48**	**2.52e-18**	**0.00e+00**	**4.90e-01**	**6.28e-02**	**4.95e-03**	**6.30e-04**	**7.46e-02**
2	**152**	**7.93e-18**	**6.20e-20**	**2.09e-01**	**2.13e-02**	**2.21e-03**	**2.15e-04**	**1.02e-02**
3	**316**	**1.33e-18**	**5.45e-21**	**1.17e-01**	**1.15e-02**	**1.20e-03**	**1.15e-04**	**2.81e-03**
4	**816**	**2.26e-18**	**2.28e-20**	**4.65e-02**	**4.49e-03**	**4.73e-04**	**4.50e-05**	**5.81e-04**
5	**908**	**2.44e-18**	**1.74e-20**	**4.17e-02**	**4.06e-03**	**4.22e-04**	**4.06e-05**	**4.34e-04**
6	**1664**	**2.03e-18**	**1.87e-20**	**2.33e-02**	**2.26e-03**	**2.35e-04**	**2.26e-05**	**1.85e-04**
7	**1928**	**2.26e-18**	**2.07e-20**	**2.00e-02**	**1.97e-03**	**2.01e-04**	**1.97e-05**	**1.47e-04**
8	**2156**	**1.54e-18**	**1.67e-20**	**1.77e-02**	**1.74e-03**	**1.77e-04**	**1.74e-05**	**1.19e-04**
9	**3368**	**1.24e-18**	**1.32e-20**	**1.15e-02**	**1.13e-03**	**1.15e-04**	**1.13e-05**	**6.68e-05**
10	**4842**	**1.10e-18**	**1.06e-20**	**7.99e-03**	**7.92e-04**	**8.00e-05**	**7.92e-06**	**3.65e-05**
11	**5730**	**8.95e-19**	**8.82e-21**	**6.63e-03**	**6.65e-04**	**6.64e-05**	**6.65e-06**	**2.71e-05**
12	**7120**	**7.83e-19**	**9.23e-21**	**5.25e-03**	**5.35e-04**	**5.26e-05**	**5.35e-06**	**1.99e-05**
13	**8348**	**7.19e-19**	**7.51e-21**	**4.39e-03**	**4.54e-04**	**4.39e-05**	**4.54e-06**	**1.52e-05**
14	**10732**	**5.35e-19**	**6.13e-21**	**3.34e-03**	**3.52e-04**	**3.35e-05**	**3.52e-06**	**1.04e-05**
15	**14486**	**5.55e-19**	**6.15e-21**	**2.53e-03**	**2.62e-04**	**2.53e-05**	**2.62e-06**	**6.94e-06**
16	**16496**	**4.98e-19**	**5.99e-21**	**2.22e-03**	**2.31e-04**	**2.22e-05**	**2.31e-06**	**5.70e-06**
17	**20604**	**4.78e-19**	**4.70e-21**	**1.75e-03**	**1.85e-04**	**1.75e-05**	**1.86e-06**	**4.14e-06**
18	**26860**	**4.09e-19**	**3.77e-21**	**1.32e-03**	**1.41e-04**	**1.32e-05**	**1.41e-06**	**2.74e-06**
19	**30440**	**3.68e-19**	**3.38e-21**	**1.15e-03**	**1.25e-04**	**1.16e-05**	**1.25e-06**	**2.22e-06**
20	**35532**	**3.13e-19**	**3.09e-21**	**9.79e-04**	**1.07e-04**	**9.80e-06**	**1.07e-06**	**1.77e-06**

Chapter 8

Conclusions

For the numerical solution of optimal control problems with distributed controls for linear second order elliptic boundary value problems we have developed, analyzed, and implemented an adaptive mixed finite element method based on the mixed formulation of the associated optimality system. We have focused on

- the iterative solution of the resulting algebraic saddle point problem by a preconditioned Richardson-type iterative scheme featuring preconditioners constructed by means of appropriately chosen left and right transforms,

- the derivation of a reliable residual-type a posteriori error estimator within the framework of a unified a posteriori error control.

Numerical results have confirmed the theoretical findings and thus documented the feasibility of the adaptive approach.

So far we have only considered the unconstrained case, i.e., we have imposed neither constraints on the control nor on the state. Future work will be devoted to the application of the adaptive approach to control constrained as well as to state constrained optimally controlled elliptic problems.

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